Appendix for *Can GDP measurement be further improved? Data revision and reconciliation*

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Appendix

A.1 Reformulating the model

To understand whether the state-space model is identified, we apply the conditions in Komunjer and Ng (2011). To follow and apply their proof, we adopt their notation for state-space models, which they write as[1]

$$X_{t+1} = A(\theta) \cdot X_t + B(\theta) \cdot \varepsilon_{t+1} \tag{A.1}$$

$$Y_{t+1} = C(\theta) \cdot X_t + D(\theta) \cdot \varepsilon_{t+1} \tag{A.2}$$

where $\theta \in \Theta$ is a real-valued parameter vector of length n_θ.

In the case of our model of GDE and GDI with l releases each, we have

$$Y_t \equiv \left[GDE_t^{1st}, GDE_t^{2nd}, \ldots, GDE_t^{l}, GDI_t^{1st}, GDI_t^{2nd}, \ldots, GDI_t^{l}\right]', \text{ a } 2l \times 1 \text{ column vector}$$

$$A(\theta) \equiv \rho, \text{ a scalar coefficient such that } 0 < |\rho| < 1$$

$$X_t \equiv \widetilde{y}_t, \text{ a scalar}$$

$$B(\theta) \equiv \left[\sigma_\nu^{E\prime}, \left(\sigma_\nu^{EI} + \sigma_\nu^{I}\right)', \mathbf{0}_{1\times 2l}\right], \text{ a } 1 \times 4l \text{ vector}$$

$$\varepsilon_t \equiv \text{a } 4l \times 1 \text{ column vector of i.i.d. random variables, jointly } \sim N(0, I)$$

$$C(\theta) \equiv \rho \cdot \iota_{2l\times 1}, \ (\iota \equiv \text{a column vector of 1's})$$

$$D(\theta) \equiv \begin{bmatrix} V \cdot \searrow\left(\sigma_\nu^{E}\right) & V \cdot \searrow\left(\sigma_\nu^{EI}\right) & \searrow\left(\sigma_\zeta^{E}\right) & \searrow\left(\sigma_\zeta^{EI}\right) \\ \mathbf{0}_{l\times l} & V \cdot \searrow\left(\sigma_\nu^{I}\right) & \mathbf{0}_{l\times l} & \searrow\left(\sigma_\zeta^{I}\right) \end{bmatrix}$$

$$V \equiv \begin{bmatrix} 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \\ \vdots & & \ddots & 0 \\ 1 & \ldots & \ldots & 1 \end{bmatrix}, \quad \begin{matrix}\text{an } l \times l \text{ matrix with 0's above the main diagonal} \\ \text{and 1's everywhere else}\end{matrix}$$

$$\searrow(\sigma) \equiv \text{a diagonal matrix with the elements of } \sigma \text{ along the main diagonal}$$

$$\theta \equiv \left[\rho, \sigma_\nu^{E\prime}, \sigma_\nu^{EI\prime}, \sigma_\nu^{I\prime}, \sigma_\zeta^{E\prime}, \sigma_\zeta^{EI\prime}, \sigma_\zeta^{I\prime}\right] \text{ and each element of } \left\{\sigma_\nu^{E}, \sigma_\nu^{I}, \sigma_\zeta^{E}, \sigma_\zeta^{I}\right\} \text{ is } > 0$$

[1] See Komunjer and Ng (2011), equations (1a) and (1b) respectively.

$\sigma_\nu^E, \sigma_\nu^{EI}, \sigma_\nu^I, \sigma_\zeta^E, \sigma_\zeta^{EI}, \sigma_\zeta^I$ are each $l \times 1$ column vectors, giving $\theta 1 + 6l$ parameters in total.

A.2 Model Assumptions

Komunjer and Ng (2011) base their identification conditions on five underlying assumptions.[2]

1. $\forall\, \theta \in \Theta, t, s$

   (a) $E(\varepsilon_t) = 0$

   (b) $E(\varepsilon_t \cdot \varepsilon_s') = \delta_{t-s} \cdot \Sigma_\varepsilon(\theta)$
   where δ_{t-s} is a real-valued scalar and $\Sigma_\varepsilon(\theta)$ is positive definite with Cholesky decomposition $L_\varepsilon(\theta)$.

   As noted above, we define $\varepsilon_t \sim N(0, I)$ and i.i.d., in which case $\Sigma_\varepsilon(\theta) = I$ and $\delta_{t-s} = 1$ for $t = s$ and 0 otherwise.

2. $\forall\, \theta \in \Theta, z \qquad |z \cdot I - A(\theta)| = 0 \Longrightarrow |z| < 1$

   In our case, $A(\theta)$ is the scalar value ρ, so this assumption reduces to $|z - \rho| = 0 \Longrightarrow z = \rho$. Therefore, the above assumption will be correct so long as $|\rho| < 1$. We have instead imposed the slightly stricter assumption that $0 < |\rho| < 1$.

3. The mapping $\theta \longrightarrow \Lambda(\theta)$ is continuously differentiable.

   As we note below, this assumption is optional and will be not required for our proof.

4. $\forall\, \theta \in \Theta \qquad D(\theta) \cdot \Sigma_\varepsilon(\theta) \cdot D(\theta)'$ is non-singular.

[2] They classify our model as "non-singular" (because the dimension of ε is greater than that of Y), so assumptions 3, 4 and 5 below refer to their assumptions for "non-singular" models.

As noted for Assumption 1, in our case $\Sigma_\varepsilon(\theta) = I$, so this reduces to

$$
\begin{aligned}
&D(\theta) \cdot D(\theta)' = \\
&\qquad \begin{bmatrix} V\cdot \searrow(\sigma_\nu^E) & V\cdot \searrow(\sigma_\nu^{EI}) & \searrow(\sigma_\zeta^E) & \searrow(\sigma_\zeta^{EI}) \\ \mathbf{0}_{l\times l} & V\cdot \searrow(\sigma_\nu^I) & \mathbf{0}_{l\times l} & \searrow(\sigma_\zeta^I) \end{bmatrix} \\
&\qquad \cdot \begin{bmatrix} V\cdot \searrow(\sigma_\nu^E) & V\cdot \searrow(\sigma_\nu^{EI}) & \searrow(\sigma_\zeta^E) & \searrow(\sigma_\zeta^{EI}) \\ \mathbf{0}_{l\times l} & V\cdot \searrow(\sigma_\nu^I) & \mathbf{0}_{l\times l} & \searrow(\sigma_\zeta^I) \end{bmatrix}' \\
&= \begin{bmatrix} V\cdot \searrow\left(\sigma_\nu^{E^2} + \sigma_\nu^{EI^2}\right)\cdot V' + \searrow\left(\sigma_\zeta^{E^2} + \sigma_\zeta^{EI^2}\right) & V\cdot \searrow(\sigma_\nu^{EI})\cdot \searrow(\sigma_\nu^I)\cdot V' + \searrow(\sigma_\zeta^{EI})\cdot \searrow(\sigma_\zeta^I) \\ V\cdot \searrow(\sigma_\nu^I)\cdot \searrow(\sigma_\nu^{EI})\cdot V' + \searrow(\sigma_\zeta^{EI})\cdot \searrow(\sigma_\zeta^I) & V\cdot \searrow(\sigma_\nu^I)\cdot \searrow(\sigma_\nu^I)\cdot V' + \searrow\left(\sigma_\zeta^{I^2}\right) \end{bmatrix} \\
&= \begin{bmatrix} V\cdot \searrow\left(\sigma_\nu^{E^2} + \sigma_\nu^{EI^2}\right)\cdot V' & V\cdot \searrow(\sigma_\nu^{EI})\cdot \searrow(\sigma_\nu^I)\cdot V' \\ V\cdot \searrow(\sigma_\nu^I)\cdot \searrow(\sigma_\nu^{EI})\cdot V' & V\cdot \searrow(\sigma_\nu^I)\cdot \searrow(\sigma_\nu^I)\cdot V' \end{bmatrix} \\
&\qquad + \begin{bmatrix} \searrow\left(\sigma_\zeta^{E^2} + \sigma_\zeta^{EI^2}\right) & \searrow(\sigma_\zeta^{EI})\cdot \searrow(\sigma_\zeta^I) \\ \searrow(\sigma_\zeta^{EI})\cdot \searrow(\sigma_\zeta^I) & \searrow\left(\sigma_\zeta^{I^2}\right) \end{bmatrix} \\
&= \Sigma_\nu + \Sigma_\zeta
\end{aligned}
$$

where Σ_ν is the $2l \times 2l$ variance-covariance matrix of the news errors and Σ_ζ is the $2l \times 2l$ variance-covariance matrix of the noise errors. Given that each element of $\{\sigma_\nu^E, \sigma_\nu^I, \sigma_\zeta^E, \sigma_\zeta^I\}$ is > 0, both Σ_ν and Σ_ζ will be Positive Definite.

5. In the case where X_t is scalar, they assume that

   (a) $K(\theta)$ has full row rank.

   We return to this assumption, below.

   (b) $C(\theta)'$ has full column rank.

   In our case $C(\theta)$ is $\rho \cdot \iota_{2l\times 1}$, so this assumption is already implied by Assumption 2.

Assumptions 3 and 5a relate to Komunjer and Ng (2011)'s reparameterisation of the system in the form[3]

$$\widehat{X}_{t+1|t+1} = A(\theta) \cdot \widehat{X}_{t|t} + K(\theta) \cdot a_{t+1} \tag{A.3}$$

$$Y_{t+1} = C(\theta) \cdot \widehat{X}_{t|t} + a_{t+1} \tag{A.4}$$

where

$$\begin{aligned}
\widehat{X}_{t|t} &\equiv \text{ the optimal linear predictor of } x_t \text{ given } \{Y_t, Y_{t-1}, \ldots\} \\
a_t &\equiv Y_{t+1} - C(\theta) \cdot \widehat{X}_{t|t}, \text{ the 1-step-ahead forecast error} \\
E(a_t \cdot a_t') &= \Sigma_a(\theta) \text{ with Cholesky decomposition } L_a(\theta) \\
K(\theta) &\equiv \text{the steady-state Kalman Gain} \\
\Lambda(\theta) &\equiv \left[\text{vec}(A(\theta))', \text{vec}(K(\theta))', \text{vec}(C(\theta))', \text{vech}(\Sigma_a(\theta))'\right]
\end{aligned}$$

They note that[4]

$$\Sigma_a(\theta) = C(\theta) \cdot \overline{\Sigma}(\theta) \cdot C(\theta)' + D(\theta) \cdot \Sigma_\varepsilon(\theta) \cdot D(\theta)' \tag{A.5}$$

$$K(\theta) = \left[A(\theta) \cdot \overline{\Sigma}(\theta) \cdot C(\theta)' + B(\theta) \cdot \Sigma_\varepsilon(\theta) \cdot D(\theta)'\right] \cdot \Sigma_a^{-1}(\theta) \tag{A.6}$$

where $\overline{\Sigma}(\theta)$ is the positive semi-definite solution to the discrete algebraic Riccati equation

$$\overline{\Sigma} = A \cdot \overline{\Sigma} \cdot A' \tag{A.7}$$

$$+ B \cdot \Sigma_\varepsilon \cdot B' - \left[A \cdot \overline{\Sigma} \cdot C' + B \cdot \Sigma_\varepsilon \cdot D'\right] \cdot \left[C \cdot \overline{\Sigma} \cdot C' + D \cdot \Sigma_\varepsilon \cdot D'\right]^{-1} \cdot \left[C \cdot \overline{\Sigma} \cdot A' + D \cdot \Sigma_\varepsilon \cdot B'\right]$$

$$= A \cdot \overline{\Sigma} \cdot A' + B \cdot \Sigma_\varepsilon \cdot B' - K \cdot \Sigma_a \cdot K' \tag{A.8}$$

(where we have temporarily dropped the dependence of all of these terms on θ to economise on notation.) We can simplify (A.5) and (A.6) somewhat using that facts that

- $A(\theta) \equiv \rho$
- $C(\theta) \equiv \rho \cdot \iota_{2l \times 1}$

[3]See their equations (9a) and (9b).

[4]See Komunjer and Ng (2011), equations (18) and (19).

- $\overline{\Sigma}(\theta)$ is scalar.[5]

- $\Sigma_\varepsilon(\theta) = I$

to obtain

$$\Sigma_a(\theta) = \overline{\Sigma}(\theta) \cdot \rho^2 \cdot 1_{2l\times 2l} + D(\theta) \cdot D(\theta)' \tag{A.9}$$

$$K(\theta) = \left[\rho^2 \cdot \overline{\Sigma}(\theta) \cdot \iota_{1\times 2l} + B(\theta) \cdot D(\theta)'\right] \cdot \Sigma_a^{-1}(\theta) \tag{A.10}$$

Beginning with (A.9), we see that $\Sigma_a(\theta)$ is the sum of a positive semi-definite matrix and a positive-definite matrix, so the result must be positive-definite (implying that the variance of the one-step-ahead forecast errors is non-zero for all elements of Y_{t+1}.) This also implies that $\Sigma_a^{-1}(\theta)$ exists. Moving to (A.10), this also implies that $K(\theta)$ has full row rank whenever $\left[\rho^2 \cdot \overline{\Sigma}(\theta) \cdot \iota_{1\times 2l} + B(\theta) \cdot D(\theta)'\right]$ has full row rank. Because this is a $1 \times 2l$ row vector, this will satisfied whenever

$$\begin{aligned} 0_{1\times 2l} &\neq \left[\rho^2 \cdot \overline{\Sigma}(\theta) \cdot \iota_{1\times 2l} + B(\theta) \cdot D(\theta)'\right] \\ \text{or } \rho^2 \cdot \overline{\Sigma}(\theta) \cdot \iota_{1\times 2l} &\neq -B(\theta) \cdot D(\theta)' \end{aligned} \tag{A.11}$$

Using the fact that

$$\begin{aligned} B \cdot D' &= \left[\sigma_\nu^{E\prime}, \left(\sigma_\nu^{EI} + \sigma_\nu^I\right)', \mathbf{0}_{1\times 2l}\right] \cdot \begin{bmatrix} V \cdot \searrow\left(\sigma_\nu^E\right) & V \cdot \searrow\left(\sigma_\nu^{EI}\right) & \searrow\left(\sigma_\zeta^E\right) & \searrow\left(\sigma_\zeta^{EI}\right) \\ \mathbf{0}_{l\times l} & V \cdot \searrow\left(\sigma_\nu^I\right) & \mathbf{0}_{l\times l} & \searrow\left(\sigma_\zeta^I\right) \end{bmatrix}' \\ &= \left[\sigma_\nu^{E\prime}, \left(\sigma_\nu^{EI} + \sigma_\nu^I\right)', \mathbf{0}_{1\times 2l}\right] \cdot \begin{bmatrix} \searrow\left(\sigma_\nu^E\right) \cdot V' & \mathbf{0}_{l\times l} \\ \searrow\left(\sigma_\nu^{EI}\right) \cdot V' & \searrow\left(\sigma_\nu^I\right) \cdot V' \\ \searrow\left(\sigma_\zeta^E\right) & \mathbf{0}_{l\times l} \\ \searrow\left(\sigma_\zeta^{EI}\right) & \searrow\left(\sigma_\zeta^I\right) \end{bmatrix} \\ &= \left[\sigma_\nu^{E\prime} \cdot \searrow\left(\sigma_\nu^E\right) \cdot V' + \left(\sigma_\nu^{EI} + \sigma_\nu^I\right)' \cdot \searrow\left(\sigma_\nu^{EI}\right) \cdot V' \quad \left(\sigma_\nu^{EI} + \sigma_\nu^I\right)' \cdot \searrow\left(\sigma_\nu^I\right) \cdot V'\right] \\ &= \left[\left(\sigma_\nu^{E^2\prime} + \sigma_\nu^{EI^2\prime} + \sigma_\nu^{I\prime} \cdot \searrow\left(\sigma_\nu^{EI}\right)\right) \cdot V' \quad \left(\sigma_\nu^{EI\prime} \cdot \searrow\left(\sigma_\nu^I\right) + \sigma_\nu^{I^2\prime}\right) \cdot V'\right] \end{aligned} \tag{A.12}$$

[5]Komunjer and Ng (2011) note that assumptions 2 and 4-NS are sufficient to ensure that a solution to the Riccati equation exists.

we can rewrite (A.11) as

$$\rho^2 \cdot \overline{\Sigma}(\theta) \cdot \iota_{1\times 2l} \neq -\left[\left(\sigma_\nu^{E^2\prime} + \sigma_\nu^{EI^2\prime} + \sigma_\nu^{I\prime} \cdot \searrow \left(\sigma_\nu^{EI}\right)\right) \cdot V' \quad \left(\sigma_\nu^{EI\prime} \cdot \searrow \left(\sigma_\nu^{I}\right) + \sigma_\nu^{I^2\prime}\right) \cdot V'\right]$$

The left-hand side implies that every element of the right hand side must be equal to $\rho^2 \cdot \overline{\Sigma}(\theta) > 0$. However, V' is a full-rank matrix which takes cumulative sums; since the cumulative sums of a non-zero constant cannot themselves be constant, the condition will always be satisfied.

Therefore, we can conclude that Assumption 5a is implied by our previous assumptions.

A.3 Conditions for Identification

Komunjer and Ng (2011) provide two alternative propositions for identification in non-singular models such as ours. The first (Proposition 1-NS) requires only that Assumptions 1, 2, 4-NS and 5-NS are respected, while the second (Proposition 2-NS) requires that all five of the Assumptions described above are respected.[6]

A.3.1 Proposition 1-NS

Komunjer and Ng (2011) show that when Assumptions 1, 2, 4-NS and 5-NS hold, then θ_0 and θ_1 are observationally equivalent if and only if there exists a full rank $n_x \times n_x$ matrix T such that

$$A(\theta_1) = T \cdot A(\theta_0) \cdot T^{-1} \tag{A.13}$$

$$C(\theta_1) = C(\theta_0) \cdot T^{-1} \tag{A.14}$$

$$K(\theta_1) = T \cdot K(\theta_0) \tag{A.15}$$

$$\Sigma_a(\theta_1) = \Sigma_a(\theta_0) \tag{A.16}$$

[6]See Komunjer and Ng (2011), p. 2008-2009.

However, the fact that $n_x = 1$ implies that T must be a non-zero real-valued scalar, while the fact that $A(\theta) = \rho$ means that (A.13) simplifies to

$$\rho_1 = T \cdot \rho_0 \cdot T^{-1} = \rho_0 \cdot T \cdot T^{-1} = \rho_0$$

This in turn implies that (A.14) simplifies to

$$\begin{aligned} \rho_1 \cdot \iota_{2l\times 1} &= \rho_0 \cdot \iota_{2l\times 1} \cdot T^{-1} \\ \therefore \rho_0 \cdot \iota_{2l\times 1} &= \rho_0 \cdot \iota_{2l\times 1} \cdot T^{-1} \\ \therefore 1 &= T \end{aligned}$$

This can be used to simplify (A.15) to

$$K(\theta_1) = K(\theta_0) \tag{A.17}$$

At this point, it is interesting to review the heuristic argument offered in the body of the paper; that by adding more vintages to our state-space model, the number of observable moments increases faster than the number of unknown parameters, thereby enabling identification of the model parameters when $l > 1$. Komunjer and Ng (2011) note via (A.3) and (A.4) that $\{A(\theta), C(\theta), K(\theta), \Sigma_a(\theta)\}$ capture all the observable moments of our series. Their Proposition 1-NS states that if these are enough to uniquely identify θ, then no two distinct values of θ will be observationally equivalent. In our case, we see that $A(\theta)$ uniquely determines $\rho \in \theta$ and that $C(\theta)$ provides no additional information on θ. That leaves $6l$ elements of θ to be identified by $K(\theta)$ and $\Sigma_a(\theta)$. $K(\theta)$ has $2l$ elements and $\Sigma_a(\theta)$ is a symmetric $2l \times 2l$ matrix which therefore has $2l \cdot (2l+1)/2 = 2l^2 + l$ distinct elements, so together they give $3l + 2l^2$ elements to identify $6l$ elements in θ. Obviously

$$3l + 2l^2 \geq 6l \quad \Longleftrightarrow \quad 2l^2 \geq 3l \quad \Longleftrightarrow \quad l \cdot (2l - 3) \geq 0$$

This is sufficient to establish that the model cannot be identified when $l = 1$, but may be identified for $l \geq 2$.

We now consider whether $\exists \;\; \theta_1 \neq \theta_0 \mid \Sigma_a(\theta_1) = \Sigma_a(\theta_0)$ and $K(\theta_1) = K(\theta_0)$ given

$\rho_1 = \rho_0 = \rho$.

A.3.2 Conditions on $\Sigma_a(\theta)$

First we note that (A.16) and (A.9) implies

$$\Sigma_a(\theta_1) = \overline{\Sigma}(\theta_1) \cdot \rho^2 \cdot 1_{2l\times 2l} + D(\theta_1) \cdot D(\theta_1)'$$
$$= \overline{\Sigma}(\theta_0) \cdot \rho^2 \cdot 1_{2l\times 2l} + D(\theta_0) \cdot D(\theta_0)' = \Sigma_a(\theta_0)$$
$$\therefore \left[\overline{\Sigma}(\theta_1) - \overline{\Sigma}(\theta_0)\right] \cdot \rho^2 \cdot 1_{2l\times 2l} = \left[D(\theta_0) \cdot D(\theta_0)'\right] - \left[D(\theta_1) \cdot D(\theta_1)'\right] \tag{A.18}$$

However, $\left[\overline{\Sigma}(\theta_1) - \overline{\Sigma}(\theta_0)\right] \cdot \rho^2$ is scalar, which implies that all $l \cdot (2l+1)$ entries of $\left[D(\theta_0) \cdot D(\theta_0)'\right] - \left[D(\theta_1) \cdot D(\theta_1)'\right]$ must be identical and equal to $\left[\overline{\Sigma}(\theta_1) - \overline{\Sigma}(\theta_0)\right] \cdot \rho^2$. Now let $[]_{ij}$ represent the element in ith row and jth column of a matrix or vector, and let $d_j(\theta) \equiv \left[\sigma_\nu^E\right]_{j1}^2 + \left[\sigma_\nu^{EI}\right]_{j1}^2$, $f_j(\theta) \equiv \left[\sigma_\nu^I\right]_{j1} \cdot \left[\sigma_\nu^{EI}\right]_{j1}$, $g_j(\theta) \equiv \left[\sigma_\nu^I\right]_{j1}^2$ so that

$$\left[D(\theta) \cdot D(\theta)'\right]_{ij} =$$
$$\left[\sigma_\zeta^E\right]_{j1}^2 + \left[\sigma_\zeta^{EI}\right]_{j1}^2 + \sum_{k=1}^{j} d_k(\theta) \quad \forall\ i,j\ |i=j,\ 1 \le j \le l \tag{A.19}$$
$$\sum_{k=1}^{\min(i,j)} d_k(\theta) \quad \forall\, i,j \mid i \ne j,\ 1 \le i,j \le l \tag{A.20}$$
$$\left[\sigma_\zeta^I\right]_{j1}^2 + \sum_{k=1}^{j} g_k(\theta) \quad \forall\, i,j \mid i=j,\ l+1 \le j \le 2l \tag{A.21}$$
$$\sum_{k=1}^{\min(i,j)} g_k(\theta) \quad \forall\, i,j \mid i \ne j,\ l+1 \le i,j \le 2l \tag{A.22}$$
$$\left[\sigma_\zeta^I\right]_{j1} \cdot \left[\sigma_\zeta^{EI}\right]_{j1} + \sum_{k=1}^{\min(i,j)} f_k(\theta) \quad \forall\, i,j \mid |i-j| = l,\ 1 \le i,j \le 2l \tag{A.23}$$
$$\sum_{k=1}^{\min(i,j)} f_k(\theta) \quad \text{otherwise} \tag{A.24}$$

Note that in the special case where $l = 1$, (A.20) (A.21) and (A.24) do not apply because it will always be the case that $i = j$. When $l > 1$, however, since (A.18) implies that all entries of $\left[D(\theta_0) \cdot D(\theta_0)'\right] - \left[D(\theta_1) \cdot D(\theta_1)'\right]$ must be identical, (A.21) and (A.22) together imply that

$$\sum_{k=1}^{j} g_k(\theta_0) - \sum_{k=1}^{j} g_k(\theta_1) = \left[\sigma_\zeta^I(\theta_0)\right]_{j1}^2 - \left[\sigma_\zeta^I(\theta_1)\right]_{j1}^2 + \sum_{k=1}^{j} g_k(\theta_0) - \sum_{k=1}^{j} g_k(\theta_1)$$
$$\therefore \sigma_\zeta^{I^2}(\theta_0) = \sigma_\zeta^{I^2}(\theta_1) \tag{A.25}$$

so the l parameters in σ_ζ^I are identified. Similar reasoning applied to (A.23) and (A.24) shows that

$$\sigma_\zeta^I(\theta_0) \cdot \sigma_\zeta^{EI}(\theta_0) = \sigma_\zeta^I(\theta_1) \cdot \sigma_\zeta^{EI}(\theta_1)$$
$$\therefore \sigma_\zeta^{EI}(\theta_0) = \sigma_\zeta^{EI}(\theta_1) \tag{A.26}$$

and analogously (A.19) and (A.20) imply

$$\sigma_\zeta^{E^2}(\theta_0) + \sigma_\zeta^{EI^2}(\theta_0) = \sigma_\zeta^{E^2}(\theta_1) + \sigma_\zeta^{EI^2}(\theta_1)$$
$$\therefore \sigma_\zeta^{E^2}(\theta_0) = \sigma_\zeta^{E^2}(\theta_1) \tag{A.27}$$

so the $2l$ parameters in σ_ζ^{EI} and σ_ζ^E are also identified whenever $l > 1$.

In addition, (A.18) and (A.20) give us

$$\left[\overline{\Sigma}(\theta_1) - \overline{\Sigma}(\theta_0)\right] \cdot \rho^2 = \sum_{j=1}^{m} d_j(\theta_0) - \sum_{j=1}^{m} d_j(\theta_1) \qquad \forall\, m \mid 1 \leq m \leq l$$

Since this holds for $m = 1$, it follows that $\left[\overline{\Sigma}(\theta_1) - \overline{\Sigma}(\theta_0)\right] \cdot \rho^2 = d_1(\theta_0) - d_1(\theta_1)$. Subtracting this from the above equation gives

$$0 = \sum_{j=2}^{m} d_j(\theta_0) - \sum_{j=2}^{m} d_j(\theta_1) \qquad \forall\, m \mid 2 \leq m \leq l$$
$$\therefore d_j(\theta_0) = d_j(\theta_1) \qquad \forall\, j \mid 2 \leq j \leq l$$
$$\therefore \left[\sigma_\nu^E(\theta_0)\right]_{j1}^2 + \left[\sigma_\nu^{EI}(\theta_0)\right]_{j1}^2 = \left[\sigma_\nu^E(\theta_1)\right]_{j1}^2 + \left[\sigma_\nu^{EI}(\theta_1)\right]_{j1}^2 \qquad \forall\, j \mid 2 \leq j \leq l$$

(A.18) and (A.24) give us

$$\left[\overline{\Sigma}(\theta_1) - \overline{\Sigma}(\theta_0)\right] \cdot \rho^2 = \sum_{j=1}^{m} f_j(\theta_0) - \sum_{j=1}^{m} f_j(\theta_1) \qquad \forall\, m \mid 1 \leq m \leq l$$

$$\therefore f_j(\theta_0) = f_j(\theta_1) \qquad \forall\, j \mid 2 \leq j \leq l$$

$$\therefore \left[\sigma_\nu^I(\theta_0)\right]_{j1} \cdot \left[\sigma_\nu^{EI}(\theta_0)\right]_{j1} = \left[\sigma_\nu^I(\theta_1)\right]_{j1} \cdot \left[\sigma_\nu^{EI}(\theta_1)\right]_{j1} \qquad \forall\, j \mid 2 \leq j \leq l$$

and (A.18) and (A.22) give us

$$\left[\overline{\Sigma}(\theta_1) - \overline{\Sigma}(\theta_0)\right] \cdot \rho^2 = \sum_{j=1}^{m} g_j(\theta_0) - \sum_{j=1}^{m} g_j(\theta_1) \qquad \forall\, m \mid 1 \leq m \leq l$$

$$\therefore g_j(\theta_0) = g_j(\theta_1) \qquad \forall\, j \mid 2 \leq j \leq l \tag{A.28}$$

$$\therefore \left[\sigma_\nu^I(\theta_0)\right]_{j1} = \left[\sigma_\nu^I(\theta_1)\right]_{j1} \qquad \forall\, j \mid 2 \leq j \leq l \tag{A.29}$$

$$\therefore \left[\sigma_\nu^{EI}(\theta_0)\right]_{j1} = \left[\sigma_\nu^{EI}(\theta_1)\right]_{j1} \qquad \forall\, j \mid 2 \leq j \leq l \tag{A.30}$$

$$\therefore \left[\sigma_\nu^E(\theta_0)\right]_{j1}^2 = \left[\sigma_\nu^E(\theta_1)\right]_{j1}^2 \qquad \forall\, j \mid 2 \leq j \leq l \tag{A.31}$$

so an additional $3 \cdot (l-1)$ parameters are also identified whenever $l > 1$. That leaves only the three parameters $\left\{\left[\sigma_\nu^I(\theta)\right]_{11}, \left[\sigma_\nu^{EI}(\theta)\right]_{11}, \left[\sigma_\nu^E(\theta)\right]_{11}\right\}$. For them, (A.18) implies

$$\left[\sigma_\nu^I(\theta_0)\right]_{11}^2 - \left[\sigma_\nu^I(\theta_1)\right]_{11}^2 = \left[\sigma_\nu^I(\theta_0)\right]_{11} \cdot \left[\sigma_\nu^{EI}(\theta_0)\right]_{11} - \left[\sigma_\nu^I(\theta_1)\right]_{11} \cdot \left[\sigma_\nu^{EI}(\theta_1)\right]_{11} \tag{A.32}$$

and

$$\left[\sigma_\nu^I(\theta_0)\right]_{11}^2 - \left[\sigma_\nu^I(\theta_1)\right]_{11}^2 = \left[\sigma_\nu^E(\theta_0)\right]_{11}^2 + \left[\sigma_\nu^{EI}(\theta_0)\right]_{11}^2 - \left[\sigma_\nu^E(\theta_1)\right]_{11}^2 - \left[\sigma_\nu^{EI}(\theta_1)\right]_{11}^2 \tag{A.33}$$

This together with any element from (A.18) involving $\left[\overline{\Sigma}(\theta_1) - \overline{\Sigma}(\theta_0)\right] \cdot \rho^2$, such as

$$\left[\overline{\Sigma}(\theta_1) - \overline{\Sigma}(\theta_0)\right] \cdot \rho^2 = \left[\sigma_\nu^I(\theta_0)\right]_{11}^2 - \left[\sigma_\nu^I(\theta_1)\right]_{11}^2 \tag{A.34}$$

gives us a system of three quadratic equations in the three remaining parameters, plus $\overline{\Sigma}(\theta)$. For the latter, we must turn to the Riccati Equation.

A.3.3 DARE

As we noted above, the Discrete Algebraic Riccati Equation may be written as

$$\overline{\Sigma} = A \cdot \overline{\Sigma} \cdot A' + B \cdot \Sigma_\varepsilon \cdot B' - K \cdot \Sigma_a \cdot K'$$

Given (A.16) and (A.17), this implies that

$$\begin{aligned}&\overline{\Sigma}(\theta_0) - A(\theta_0) \cdot \overline{\Sigma}(\theta_0) \cdot A(\theta_0)' - B(\theta_0) \cdot \Sigma_\varepsilon(\theta_0) \cdot B(\theta_0)' \\ &= \overline{\Sigma}(\theta_1) - A(\theta_1) \cdot \overline{\Sigma}(\theta_1) \cdot A(\theta_1)' - B(\theta_1) \cdot \Sigma_\varepsilon(\theta_1) \cdot B(\theta_1)'\end{aligned}$$

which we may simplify using $\Sigma_\varepsilon = I$ and $A = \rho$ to give

$$\begin{aligned}\overline{\Sigma}(\theta_0) \cdot \left(1-\rho^2\right) - B(\theta_0) \cdot B(\theta_0)' &= \overline{\Sigma}(\theta_1) \cdot \left(1-\rho^2\right) - B(\theta_1) \cdot B(\theta_1)' \\ \therefore \left[\overline{\Sigma}(\theta_0) - \overline{\Sigma}(\theta_1)\right] \cdot \left(1-\rho^2\right) &= B(\theta_0) \cdot B(\theta_0)' - B(\theta_1) \cdot B(\theta_1)' \\ &= \sum_{j=1}^{l} \left[\sigma_\nu^E(\theta_0)\right]_{j1}^2 + (\left[\sigma_\nu^{EI}(\theta_0)\right]_{j1} + \left[\sigma_\nu^I(\theta_0)\right]_{j1})^2 \\ &- \sum_{j=1}^{l} \left[\sigma_\nu^E(\theta_1)\right]_{j1}^2 + (\left[\sigma_\nu^{EI}(\theta_1)\right]_{j1} + \left[\sigma_\nu^I(\theta_1)\right]_{j1})^2\end{aligned}$$

However, from the results above, we know that $\forall\ j \mid 2 \leq j \leq l$ the terms in the two summations will be identical, so this simplifies further to

$$\begin{aligned}&\left[\overline{\Sigma}(\theta_0) - \overline{\Sigma}(\theta_1)\right] \cdot \left(1-\rho^2\right) = \\ &\left(\left[\sigma_\nu^E(\theta_0)\right]_{11}^2 - \left[\sigma_\nu^E(\theta_1)\right]_{11}^2\right) + (\left[\sigma_\nu^{EI}(\theta_0)\right]_{11} + \left[\sigma_\nu^I(\theta_0)\right]_{11})^2 - (\left[\sigma_\nu^{EI}(\theta_1)\right]_{11} + \left[\sigma_\nu^I(\theta_1)\right]_{11})^2\end{aligned}$$

We may use this to eliminate $\left[\overline{\Sigma}(\theta_0) - \overline{\Sigma}(\theta_1)\right]$ from (A.34) and write our system of three equations in three unknowns more compactly using the notation $s_{i,j} \equiv [\sigma_\nu^j(\theta_i)]_{11}$ for $j \in$

$\{I, EI, E\}, i \in \{0, 1\}$ as

$$s_{0,I}^2 - s_{1,I}^2 = s_{0,I} \cdot s_{0,EI} - s_{1,I} \cdot s_{1,EI} \tag{A.35}$$

$$s_{0,I}^2 - s_{1,I}^2 = s_{0,E}^2 - s_{1,E}^2 + s_{0,EI}^2 - s_{1,EI}^2 \tag{A.36}$$

$$s_{0,I}^2 - s_{1,I}^2 = \frac{\rho^2}{\rho^2 - 1} \cdot \left[\left(s_{0,E}^2 - s_{1,E}^2\right) + \left(s_{0,EI} + s_{0,I}\right)^2 - \left(s_{1,EI} + s_{1,I}\right)^2\right] \tag{A.37}$$

(A.36) and (A.37) imply

$$s_{0,I}^2 - s_{1,I}^2 = \frac{\rho^2}{\rho^2 - 1} \cdot \left[\left(\left(s_{0,I}^2 - s_{1,I}^2\right) - \left(s_{0,EI}^2 - s_{1,EI}^2\right)\right) + \left(s_{0,EI} + s_{0.I}\right)^2 - \left(s_{1,EI} + s_{1,I}\right)^2\right]$$
$$\therefore \left(s_{0,I}^2 - s_{1,I}^2\right) \cdot \left(\frac{\rho^2 - 1}{\rho^2} - 1\right) + \left(s_{0,EI}^2 - s_{1,EI}^2\right)$$
$$= \left(s_{0,EI}^2 - s_{1,EI}^2\right) + 2 \cdot \left(s_{0,I} \cdot s_{0,EI} - s_{1,I} \cdot s_{1,EI}\right) + \left(s_{0,I}^2 - s_{1,I}^2\right)$$
$$\therefore \left(s_{0,I}^2 - s_{1,I}^2\right) \cdot \left(\frac{\rho^2 - 1}{\rho^2} - 2\right) = 2 \cdot \left(s_{0,I} \cdot s_{0,EI} - s_{1,I} \cdot s_{1,EI}\right)$$

Combing this with (A.35) gives us

$$s_{0,I} \cdot s_{0,EI} - s_{1,I} \cdot s_{1,EI} = \kappa \cdot \left(s_{0,I} \cdot s_{0,EI} - s_{1,I} \cdot s_{1,EI}\right)$$
$$\therefore 0 = (\kappa - 1) \cdot \left(s_{0,I} \cdot s_{0,EI} - s_{1,I} \cdot s_{1,EI}\right)$$

where $\kappa = 2 \cdot \left(\frac{\rho^2 - 1}{\rho^2} - 2\right)^{-1} = 2 \cdot \left(\frac{-\rho^2}{\rho^2 + 1}\right)$. Since ρ is real, this ensures that $\kappa \leq 0$, so $\kappa - 1 \neq 0$. The only solution is therefore

$$s_{0,I} \cdot s_{0,EI} = s_{1,I} \cdot s_{1,EI}$$
$$\therefore s_{0,I}^2 = s_{1,I}^2 \; ,$$
$$s_{0,EI} = s_{1,EI} \; ,$$
$$s_{0,E} = s_{1,E}$$

which implies that all the remaining model parameters are identified.

A.4 Posteriors for model with two releases of GDE and GDI

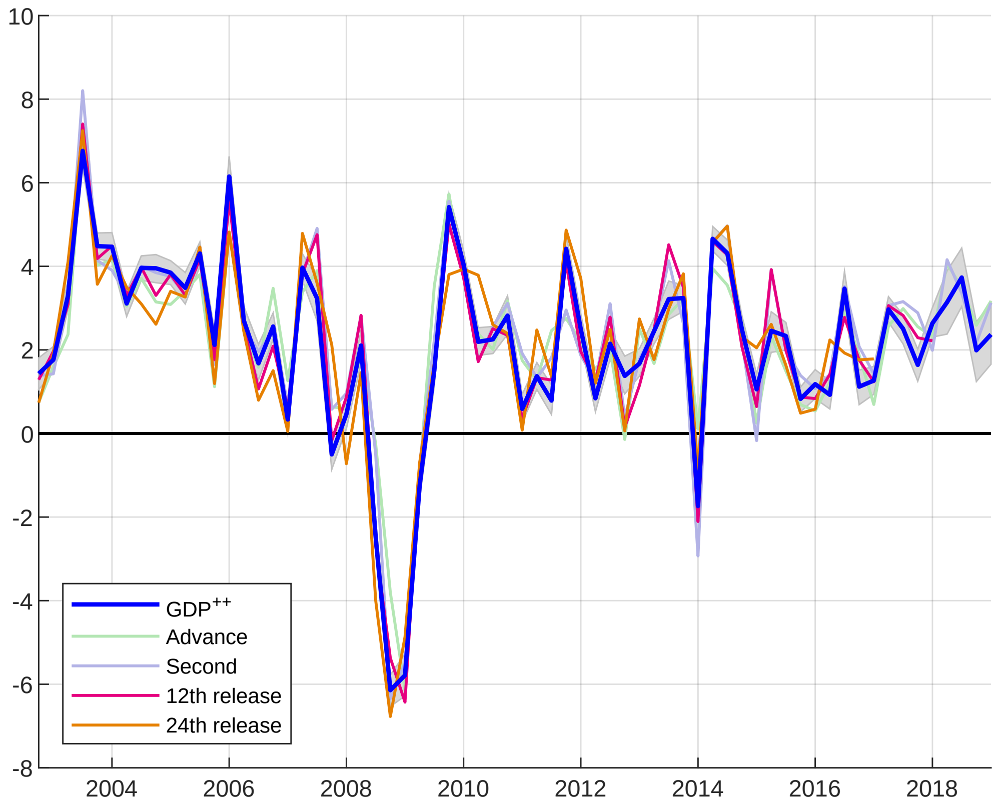
Table A.1: Posterior of parameters

	5%ile	Median	95%ile
GDP^{++}			
μ	0.2262	0.8004	1.4109
ρ	0.4161	0.6144	0.8056
$\boldsymbol{\Omega}$			
$\omega_{1,1}$	2.8586	3.9841	5.8407
$\omega_{2,1}$	-0.4416	0.0004	0.1389
$\omega_{3,1}$	-2.2235	-1.4150	-0.8998
$\omega_{4,1}$	-1.4529	-0.8078	-0.4071
$\omega_{2,2}$	0.0151	0.0427	0.1539
$\omega_{3,2}$	-0.2435	-0.0219	0.3145
$\omega_{4,2}$	-0.2435	-0.0219	0.3145
$\omega_{3,3}$	0.7785	1.2839	1.7593
$\omega_{4,3}$	0.5493	0.8185	1.2235
$\omega_{4,4}$	0.5493	0.8185	1.2235
$\omega_{5,5}$	1.0273	1.4428	2.0636
$\omega_{8,5}$	0.0842	0.3310	0.6571
$\omega_{6,6}$	0.7643	1.1643	1.7481
$\omega_{9,6}$	-0.6153	-0.4333	-0.2707
$\omega_{7,7}$	0.0120	0.0272	0.2144
$\omega_{8,8}$	0.0416	0.1498	0.4981
$\omega_{9,9}$	0.0782	0.1940	0.3602

Posterior distribution of parameters, where the left columns reports the 5th percentile, the middle column the 50th percentile and the right column the 95th percentile of the posterior distributions. The estimated model corresponds to (10) and (11) in the main body of the paper. ρ denotes the AR coefficient in the state equation of GDP^{++}, where we have added an intercept μ. Ω is the variance-covariance matrix with $\Omega = RR'$ and $\omega_{i,j}$ denoting the free parameters of Ω.

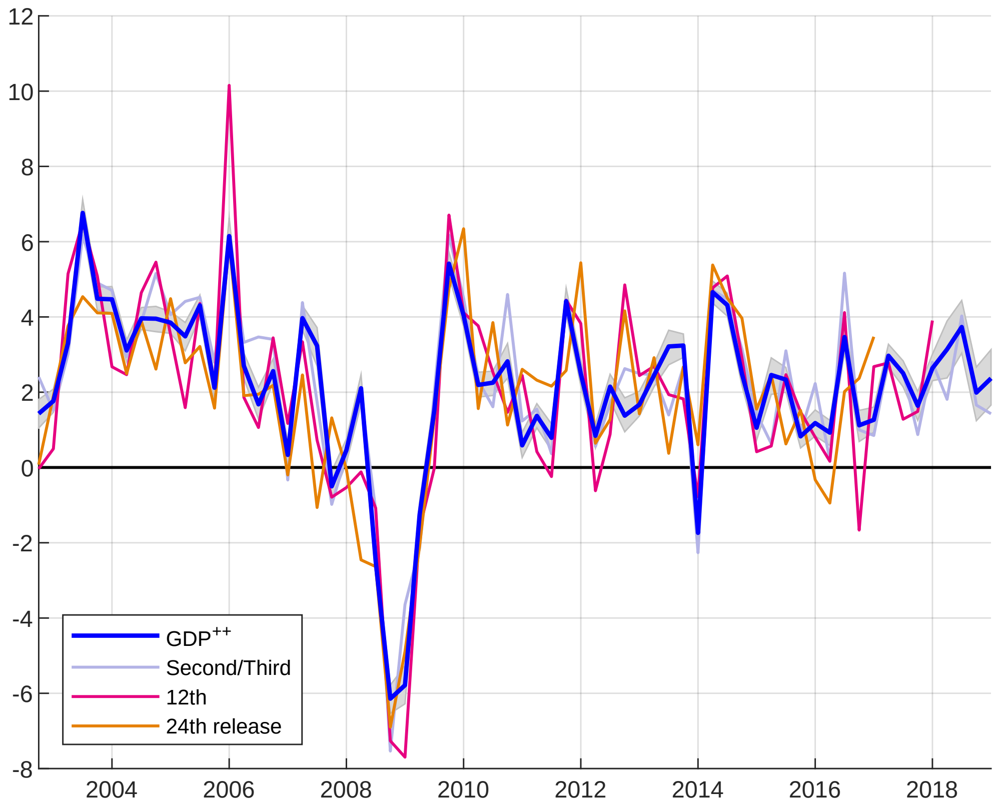
A.5 Additional figures

Figure A.1: GDP^{++} vs. GDE with "Ragged-Edge" Data



The blue line represents the posterior median of GDP^{++} (the "true" value) and the shaded area around the blue line indicates 90% of posterior probability mass. The green line represents the advance estimate, the purple line is the second estimate, the red line the 12th release and the orange line the 24th release of expenditure side GDP growth. We have incorporated a missing observations approach as described in, e.g., Durbin and Koopman (2001) to cope with ragged edges at the end of the sample.

Figure A.2: GDP^{++} vs. GDI with "Ragged-Edge" Data



The blue line represents the posterior median of GDP^{++}, the "true" value, and the shaded area around the blue line indicates 90% of posterior probability mass. The purple line is the second/third estimate, the red line the 12th release and the orange line the 24th release of income side GDP growth. We have incorporated a missing observations approach as described in, e.g., Durbin and Koopman (2001) to cope with ragged edges at the end of the sample.

References

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