Can GDP measurement be further improved? Data revision and reconciliation

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Abstract

Recent years have seen many attempts to combine expenditure-side estimates of U.S. real output (GDE) growth with income-side estimates (GDI) to improve estimates of real GDP growth. We show how to incorporate information from multiple releases of noisy data to provide more precise estimates while avoiding some of the identifying assumptions required in earlier work. This relies on a new insight: using multiple data releases allows us to distinguish news and noise measurement errors in situations where a single vintage does not. We find that (a) the data prefer averaging across multiple releases instead of discarding early releases in favour of later ones, and (b) that initial estimates of GDI are quite informative. Our new measure, GDP^{++}, undergoes smaller revisions and tracks expenditure measures of GDP growth more closely than either the simple average of the expenditure and income measures published by the BEA or the GDP growth measure of Aruoba et al. (2016) published by the Federal Reserve Bank of Philadelphia.

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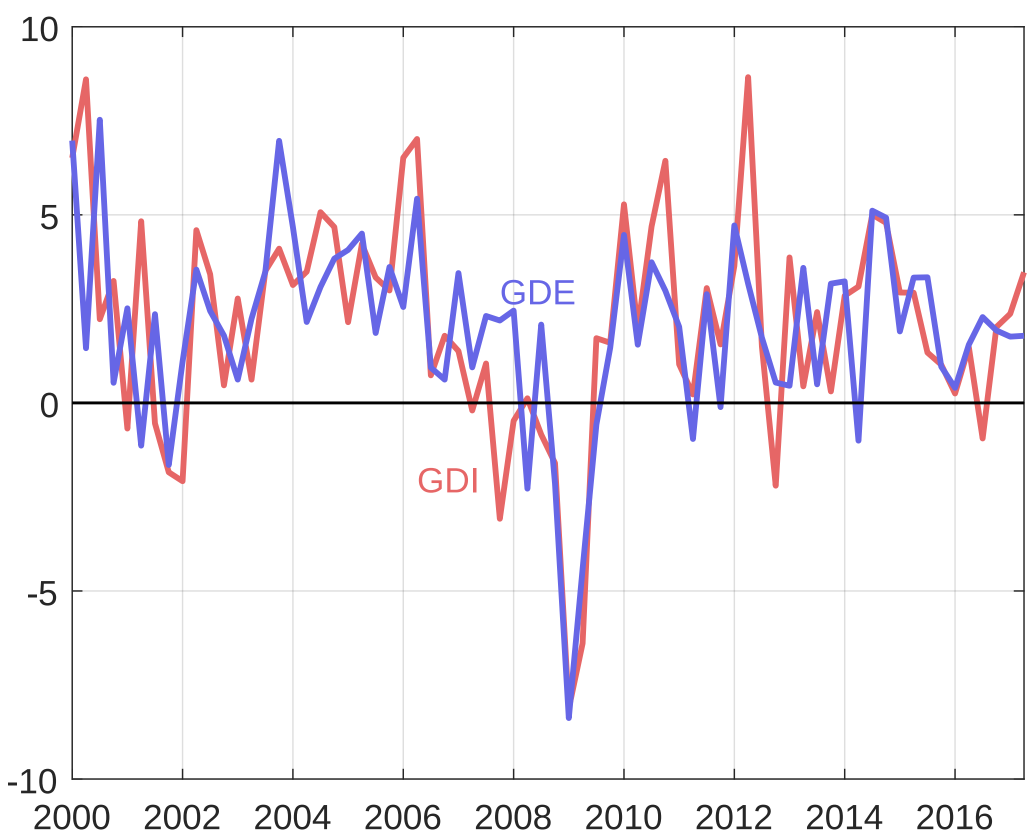
1 Introduction

Unlike many other nations, U.S. national accounts feature distinct estimates of real output growth based on the expenditure approach (GDE) and the income approach (GDI), see Figure 1. As pointed out by Stone, Champernowne and Meade (1942), while in theory these two approaches should give identical estimates, measurement errors cause discrepancies to arise.[1] These discrepancies are sometimes important. Chang and Li (2015) examine the impact of using GDI rather than GDE in nearly two dozen recent empirical papers published in major economic journals; they find substantive differences in roughly 15% of them. Nalewaik (2012) finds that GDI leads to quicker detection of U.S. recessions than GDE.

Figure 1: U.S. *GDP* growth: Expenditure side vs. income side



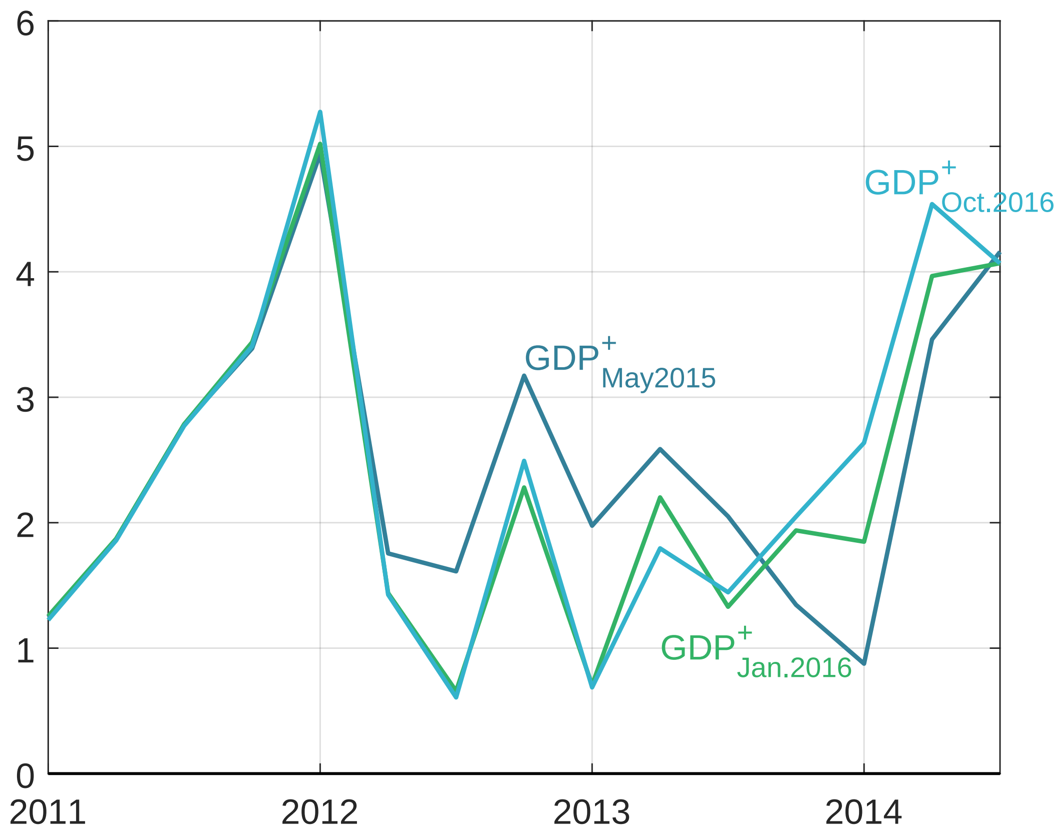
[1]The same applies to the production-based estimate of output. See, e.g., the study of Rees, Lancaster and Finlay (2015) on Australian GDP.

While several studies have tried to determine which measure should be preferred in various contexts, Weale (1992) and Diebold (2010) argue that reconciling them is a more useful response as it should incorporate more information. Fixler and Nalewaik (2009) point out, however, that reconciliation traditionally relies on the assumption that measurement errors are "noise", which in turn forces the reconciled estimate of the latent variable ("true" GDP in this case) to be less variable than any of the individual series being reconciled. They instead propose that measurement errors may also include a "news" component. While this causes a loss of identification, they glean information from the revision of GDE and GDI to place bounds on relative contributions of news and noise in a least-squares framework. Aruoba et al. (2012) consider the problem from a forecast combination perspective, assuming "news" errors and imposing priors in lieu of identification without revisions, while Aruoba et al. (2016) consider alternative identifying assumptions and propose the addition of an instrumental variable. Almuzara et al. (2018) investigate a dynamic factor model (DFM) with cointegration restrictions while Anesti et al. (2018) propose a mixed-frequency Release-Augmented DFM.

Aruoba et al. (2016) is the basis for the GDP^+ measure published by the Federal Reserve Bank of Philadelphia.[2] However, while their approach ignores the possibility of data revision, Figure 2 shows that the published series is subject to important revisions, which complicates its interpretation and use in policy decisions. Separately, Jacobs and van Norden (2011) and Kishor and Koenig (2012) propose state-space frameworks that allow estimation of both news- and noise-type measurement errors in data revision, but do not consider problems of data reconciliation. In this paper, we extend Jacobs and van Norden (2011, henceforth JvN) to consider the problem of reconciliation and identification in which there are multiple estimates of the common underlying variable, all of which are subject to revision. Allowing for both news and noise measurement errors, the result is a modeling framework substantially more general than those previously proposed. We show that identification of these two types of measurement errors is made possible by modeling data revisions as well as the dynamics of the series. We provide a historical decomposition of GDE and GDI into news and noise shocks, and we compare those series to our improved

[2] See `http://www.philadelphiafed.org/research-and-data/real-time-center/gdpplus/`

Figure 2: GDP^{+} in real-time



Various vintages of GDP^{+}. Source: Federal Reserve Bank of Philadelphia.

GDP estimate, GDP^{++}. We find that GDP^{+} is more tightly correlated with GDI releases than with the GDE corresponding releases, while the opposite is usually true for GDP^{++}. We also find that regardless of the series and the release chosen, GDP^{++} is almost always more positively correlated with the published series than GDP^{+}. Consistent with Fixler and Nalewaik (2009), noise errors seems to have a relatively more important role in GDI than in GDE, but GDI still appears to contain valuable information about output growth, particularly in its initial release.

The paper is structured as follows. In Section 2 we present our econometric framework. Comparing our model model to that of Aruoba et al. (2016), we note the incorporation of multiple data vintages increases the number of observable moments enough to provide identification of all the model's parameters whenever more than one data vintage is used. In an Appendix, we provide a detailed proof of identification based on the work of Komunjer and Ng (2011). In Section 3, we describe our data and estimation method. Results are shown in Section 4 and Section 5 concludes.

2 Econometric Framework

In this section, after establishing some notation, we describe our econometric framework. We then compare the results to the GDP^{+} model of Aruoba et al. (2016) and discuss how the identification of news and noise measurement errors differs in the two models.

We follow the standard notation in this literature by letting y_t^{t+j} be an estimate published at time $t+j$ of some real-valued scalar variable y at time t, while we follow JvN and denote the unobserved "true" value as $\tilde{y}_t$. We define $\boldsymbol{y}_t$ as a $l \times 1$ vector of l different vintage estimates of y_t^{t+i}, $i = 1, \ldots, l$ so $\boldsymbol{y}_t \equiv \left[y_t^{t+1}, y_t^{t+2}, \ldots, y_t^{t+l}\right]'$. We may stack two such series of estimates in a $2l \times 1$ vector $\boldsymbol{Y}_t \equiv \left[\boldsymbol{y}_{1,t}', \boldsymbol{y}_{2,t}'\right]' \equiv \left[y_{1,t}^{t+1}, y_{1,t}^{t+2} \ldots, y_{1,t}^{t+l}, y_{2,t}^{t+1}, y_{2,t}^{t+2}, \ldots, y_{2,t}^{t+l}\right]'$.

For state-space models, we follow the notation of Durbin and Koopman (2001)

$$\boldsymbol{Y}_t = \boldsymbol{Z} \cdot \boldsymbol{\alpha}_t + \boldsymbol{\varepsilon}_t \tag{1}$$

$$\boldsymbol{\alpha}_{t+1} = \boldsymbol{T} \cdot \boldsymbol{\alpha}_t + \boldsymbol{R} \cdot \boldsymbol{\eta}_{t+1}, \tag{2}$$

where $\boldsymbol{y}_t$ is $2l \times 1$, $\boldsymbol{\alpha}_t$ is $m \times 1$, $\boldsymbol{\varepsilon}_t$ is $2l \times 1$ and $\boldsymbol{\eta}_t$ is $r \times 1$; $\boldsymbol{\varepsilon}_t \sim N(\mathbf{0}, \boldsymbol{H})$ and $\boldsymbol{\eta}_t \sim N(\mathbf{0}, \boldsymbol{I}_r)$. Durbin and Koopman (2001) assume that both error terms are i.i.d. and orthogonal to one another.[3] However, we write our model in the form where $\boldsymbol{\varepsilon}_t$ is equal to zero and may be omitted, so (1) simplifies to

$$\boldsymbol{Y}_t = \boldsymbol{Z} \cdot \boldsymbol{\alpha}_t. \tag{3}$$

2.1 A Model for Data Reconciliation

Measurement errors are said to be noise $\left(\zeta_t^{t+i}\right)$ when they are orthogonal to the true values $\tilde{y}_t$, so that

$$y_t^{t+i} = \tilde{y}_t + \zeta_t^{t+i}, \qquad \text{cov}(\tilde{y}_t, \zeta_t^{t+i}) = 0 \qquad \forall\ i. \tag{4}$$

Noise implies that revisions $(y_t^{t+i+1} - y_t^{t+i})$ are generally forecastable. In contrast, measurement errors are described as news (ν_t^{t+i}) if and only if

$$\tilde{y}_t = y_t^{t+i} + \nu_t^{t+i}, \qquad \text{cov}(y_t^{t+j}, \nu_t^{t+i}) = 0 \qquad \forall\ j \leq i. \tag{5}$$

If data revisions are pure news errors, current and past vintages of the series will be of no use in forecasting future data revision. Various authors, such as Croushore and Stark (2001), have found that U.S. macroeconomic series seem to be neither pure news nor pure noise. We therefore allow for both types of measurement errors by partitioning the state vector $\boldsymbol{\alpha}_t$ into three components [4]

$$\boldsymbol{\alpha}_t = [\tilde{y}_t, \boldsymbol{\nu}_t', \boldsymbol{\zeta}_t']', \tag{6}$$

of length 1, $2l$ and $2l$ respectively, and we similarly partition

$$\boldsymbol{Z} = [\boldsymbol{Z}_1, \boldsymbol{Z}_2, \boldsymbol{Z}_3], \tag{7}$$

[3] For more detailed assumptions, see Durbin and Koopman (2001, Section 3.1 and 4.1.) For convenience we omit constants from the model in this exposition.

[4] JvN include a fourth component that both allows for more complex dynamics in $\tilde{y}_t$ and permits extensions to problems such as detrending and seasonal adjustment when working with data in levels. Instead, we work with growth rates and follow Aruoba et al. (2016) and many others by assuming that real output growth simply follows an AR(1) process.

where $\boldsymbol{Z}_1 = \boldsymbol{\iota}_{2l}$(a $2l$ vector of ones), $\boldsymbol{Z}_3 = \boldsymbol{Z}_4 = \boldsymbol{I}_{2l}$ (both are $2l \times 2l$ identity matrices). The measurement equation (3) therefore simplifies to

$$\boldsymbol{Y}_t = \boldsymbol{Z} \cdot \boldsymbol{\alpha}_t = \tilde{y}_t + \boldsymbol{\nu}_t + \boldsymbol{\zeta}_t = \text{`Truth'} + \text{`News'} + \text{`Noise'}.$$

Turning to the transition equation (2), the matrix $\boldsymbol{T}$ is a $(1+4l) \times (1+4l)$ matrix with only a single non-zero element ρ

$$\boldsymbol{T} = \begin{bmatrix} \rho & 0 & \dots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & 0 \end{bmatrix}, \tag{8}$$

which captures the first-order autocorrelation of the true values $\tilde{y}_t$. Both the News and the Noise measurement errors are assumed to be uncorrelated through time. What distinguishes them is how they vary across data vintages. This is determined by $\boldsymbol{R}$, a $(1+4l) \times 4l$ matrix of the form[5]

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{R}_1 & \boldsymbol{R}_2 + \boldsymbol{R}_3 & \mathbf{0} & \mathbf{0} \\ -\boldsymbol{V}_l \cdot \text{diag}(\boldsymbol{R}_1) & -\boldsymbol{V}_l \cdot \text{diag}(\boldsymbol{R}_3) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{V}_l \cdot \text{diag}(\boldsymbol{R}_2) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{R}_4 & \boldsymbol{R}_6 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{R}_5 \end{bmatrix}, \tag{9}$$

where the row vector $\boldsymbol{R}_1 = \left[\sigma_{\nu_1^1}, \sigma_{\nu_2^1}, \dots, \sigma_{\nu_l^1}\right]$ corresponds to the news uniquely found in y_1 while $\boldsymbol{R}_2 = \left[\sigma_{\nu_1^2}, \sigma_{\nu_2^2}, \dots, \sigma_{\nu_l^2}\right]$ corresponds to all the news found in y_2 and $\boldsymbol{R}_3 = \left[\sigma_{\nu_1^3}, \sigma_{\nu_3^2}, \dots, \sigma_{\nu_l^3}\right]$ captures the news common to both series. diag($\boldsymbol{R}_i$) is an $l \times l$ diagonal matrix with the elements of $\boldsymbol{R}_i$ on its main diagonal and V_l is an $l \times l$ matrix with ones above the main diagonal and zeros everywhere else. Next, $\boldsymbol{R}_4, \boldsymbol{R}_5$ and $\boldsymbol{R}_6$ are each $l \times l$ diagonal matrices with non-zero elements $\left[\sigma_{\zeta_1^i}, \sigma_{\zeta_2^i}, \dots, \sigma_{\zeta_l^i}\right]$ for $i = 4, 5, 6$. Noise errors unique to series 1 enter via $\boldsymbol{R}_4$, $\boldsymbol{R}_5$ controls all the noise in series 2, while noise errors common to both enter via $\boldsymbol{R}_6$. Finally, we partition the $4 \times l$ vector $\boldsymbol{\eta}$ into four $l \times 1$

[5] An earlier version of this paper required news and noise measurement errors to be independent across series. We would like to thank two anonymous referees for encouraging us to generalize the model.

vectors $\boldsymbol{\eta}_t = \left[\boldsymbol{\eta}'_{\nu_1 t},\ \boldsymbol{\eta}'_{\nu_2 t},\ \boldsymbol{\eta}'_{\zeta_1 t},\ \boldsymbol{\eta}'_{\zeta_2 t}\right]'$, where $\boldsymbol{\eta}_{\nu i t}$ and $\boldsymbol{\eta}_{\zeta i t}$ are the sources for news and noise measurement errors in variable i.

To illustrate, consider the following very simple case. Let $y_1 \equiv GDE$ (the growth rate of real gross domestic expenditure), $y_2 \equiv GDI$ (the growth rate of real gross domestic income) and let $l = 2$ (we only consider two vintages, the 1st and 2nd releases). Then (3) becomes

$$\begin{aligned}
\begin{bmatrix} GDE_t^{1st} \\ GDE_t^{2nd} \\ GDI_t^{1st} \\ GDI_t^{2nd} \end{bmatrix} &= \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \tilde{y}_t \\ \boldsymbol{\nu}_t \\ \boldsymbol{\zeta}_t \end{bmatrix} \\
&= \begin{bmatrix} \tilde{y}_t \\ \tilde{y}_t \\ \tilde{y}_t \\ \tilde{y}_t \end{bmatrix} + \begin{bmatrix} \nu_t^{GDE,1} & 0 & 0 & 0 \\ 0 & \nu_t^{GDE,2} & 0 & 0 \\ 0 & 0 & \nu_t^{GDI,1} & 0 \\ 0 & 0 & 0 & \nu_t^{GDI,2} \end{bmatrix} + \begin{bmatrix} \zeta_t^{GDE,1} & 0 & 0 & 0 \\ 0 & \zeta_t^{GDE,2} & 0 & 0 \\ 0 & 0 & \zeta_t^{GDI,1} & 0 \\ 0 & 0 & 0 & \zeta_t^{GDI,2} \end{bmatrix} \\
&= \text{`Truth'} + \text{`News'} + \text{`Noise'}.
\end{aligned}$$

and (2) becomes

$$\begin{bmatrix} \tilde{y}_{t+1} \\ \boldsymbol{\nu}_{t+1} \\ \boldsymbol{\zeta}_{t+1} \end{bmatrix} = \begin{bmatrix} \rho & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \tilde{y}_t \\ \boldsymbol{\nu}_t \\ \boldsymbol{\zeta}_t \end{bmatrix} + \boldsymbol{R} \cdot \boldsymbol{\eta}_{t+1},$$

where

$$\boldsymbol{R} = \begin{bmatrix} \sigma_\nu^{GDE,1} & \sigma_\nu^{GDE,2} & \sigma_\nu^{GDI,1} + \sigma_\nu^{Both,1} & \sigma_\nu^{GDI,2} + \sigma_\nu^{Both,2} & 0 & 0 & 0 & 0 \\ 0 & -\sigma_\nu^{GDE,2} & 0 & -\sigma_\nu^{Both,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sigma_\nu^{GDI,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_\zeta^{GDE,1} & 0 & \sigma_\zeta^{Both,1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_\zeta^{GDE,2} & 0 & \sigma_\zeta^{Both,2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma_\zeta^{GDI,1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_\zeta^{GDI,2} \end{bmatrix}$$

$$\boldsymbol{\eta}_{t+1} = \left[\eta_{\nu_{t+1}}^{GDE,1}, \eta_{\nu_{t+1}}^{GDE,2}, \eta_{\nu_{t+1}}^{GDI,1}, \eta_{\nu_{t+1}}^{GDI,2}, \eta_{\zeta_{t+1}}^{GDE,1}, \eta_{\zeta_{t+1}}^{GDE,2}, \eta_{\zeta_{t+1}}^{GDI,1}, \eta_{\zeta_{t+1}}^{GDI,2}\right]'.$$

2.2 Identification and GDP^+

Aruoba et al. (2016) consider the problem of identification in a special case of the GDE/GDI example considered above where only a single vintage is available ($l = 1$). Their unrestricted model may be written as[6]

$$\begin{bmatrix} GDE_t \\ GDI_t \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \tilde{y}_t \\ \epsilon_t^E \\ \epsilon_t^I \end{bmatrix} \tag{10}$$

$$\begin{bmatrix} \tilde{y}_{t+1} \\ \epsilon_{t+1}^E \\ \epsilon_{t+1}^I \end{bmatrix} = \begin{bmatrix} \rho & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \tilde{y}_t \\ \epsilon_t^E \\ \epsilon_t^I \end{bmatrix} + \begin{bmatrix} \sigma_{yy} & \sigma_{yE} & \sigma_{yI} \\ \sigma_{Ey} & \sigma_{EE} & \sigma_{EI} \\ \sigma_{Iy} & \sigma_{IE} & \sigma_{II} \end{bmatrix} \cdot \begin{bmatrix} u_{t+1}^y \\ u_{t+1}^E \\ u_{t+1}^I \end{bmatrix}. \tag{11}$$

Their $\boldsymbol{Z}$ and $\boldsymbol{T}$ matrices are identical to the comparable matrices in our model when $l = 1$ and they similarly partition the state vector into "true" values and two types of measurement errors ϵ_t^E and ϵ_t^I. However, their measurement errors are unrestricted combinations of three reduced-form errors u_t^i, $i = \{y, E, I\}$ and as such are not identified. They propose

[6] See Aruoba et al. (2016), equations (A.1) and (A.2). Their model further differs from the model above in that they model only the *sum* of news and noise shocks.

adding a third (instrumental) variable which is correlated with $\widetilde{y}_t$ but not with ϵ_t^E or ϵ_t^I, suggesting that household survey data may be suitable for this purpose. We show that the model may instead be identified by increasing the number of vintages analysed and assuming that measurement errors are the sum of news and noise measurement errors as characterised above. In the Appendix, we provide a rigorous proof of identification. In the remainder of this section, we provide a more informal argument.

Some insight comes from the form of the $\boldsymbol{R}$ matrix in (9). News and noise measurement errors have tightly constrained behaviour across successive data vintages, even when they may be correlated across series. Noise errors are assumed to be both uncorrelated across vintages and with innovations in true values. However, news errors must be correlated with one another, and with innovations in true values, and their variances must be decreasing as series are revised.

If we have two series to reconcile (such as GDE and GDI) and l vintages of each, we have $2l \cdot (2l + 1)/2$ observable cross moments as well as $2 \cdot l$ first-order autocorrelation coefficients, for a total of $l \cdot (2 \cdot l + 3)$ moments. The only free parameters in the above model, however, are the autocorrelation coefficient ρ and the $6 \cdot l$ non-zero elements of $\boldsymbol{R}$. Obviously, the number of available moments increases with l^2 while the number of free parameters increases only linearly with l.[7]

In the special case where we use only a single data release, $l = 1$, we have $1+6\cdot l = 7$ free parameters to estimate, but only $1 \cdot (2 \cdot 1 + 3) = 5$ available moments with which to do so. This is consistent with the lack of identification noted by Aruoba et al. (2016). However, if we use $l = 2$ data vintages, we have $1 + 6 \cdot 2 = 13$ free parameters and $2 \cdot (2 \cdot 2 + 3) = 14$ moments with which to identify them. For $l = 3$ we have 27 moments with which to estimate 19 parameters and for $l = 4$ (the case we consider below) we have 44 moments with which to estimate 25 parameters.[8]

It may also be useful to understand intuitively how the use of multiple vintages aids identification. Estimating true values requires us to distinguish variation due to news from that due to noise. In our model, increasing the variance of noise associated with a particular

[7]One must also keep in mind that identification by data revision requires that the data are in fact revised. If not, we effectively return to the underidentified case of $l = 1$.

[8]In the Appendix, we show that precisely the same moment conditions arise out of the restrictions derived by Komunjer and Ng (2011).

vintage unambiguously lowers the correlation of that vintage with other vintages of the same series. However, increasing the variance of its news lowers its correlation with earlier vintages but increases its correlation with later vintages *ceteris paribus*. Furthermore, in the presence of serial correlation (ρ), increasing news should increase the correlation across time while increasing noise should not. Both of these effects are present regardless of whether or not measurement errors are correlated across the series being reconciled.

Before turning to examine the usefulness of such a framework for reconciling U.S. GDE and GDI, we note that the above model could be generalized further along lines suggested by the univariate model of JvN. For example,

1. We may wish to relax some of the zero restrictions on the transition matrix in (2) to allow for measurement errors to be correlated across calendar periods. (JvN refer to these as "spillover" effects.) For example, the annual incorporation of tax return data into the National Accounts may cause revisions that are correlated across the various quarters of the tax year.

2. We may wish to allow for more than 2 alternative measures of the same underlying concept. For example, the Office of National Statistics in the UK produces real GDP estimates based on expenditure data, income data, and output data, suggesting a 3-way reconciliation.[9]

3. It may be useful to work with levels of GDP and simultaneously decompose the unobserved "true" values further, for example into seasonal and non-seasonal components, or into trend and cycle.

We leave such extensions to future research.

[9]Rees, Lancaster and Finlay (2015) explore such a reconciliation for Australian GDP using state-space models similar to those of Aruoba et al. (2012) and Aruoba et al. (2016).

3 Data and Estimation

3.1 Data

We use monthly vintages of quarterly expenditure-based and income-based estimates of GDP from the Bureau of Economic Analysis (BEA) covering the period 2002Q4–2017Q1. For GDE we employ the Advance, the Third, the 12th and the 24th releases. For GDI we take the Second/Third, the 12th and the 24th releases for GDI. (Due to a lag in source data availability the first available estimate for GDI is released at the time of the Second GDE estimate, except for the estimate of 4th quarter GDI, which is released at the time of the Third GDE estimate.[10])

3.2 Estimation

We employ Gibbs Sampling methods (see, e.g., Kim and Nelson 1999) to obtain posterior simulations for our model's parameters in (8) and (9). We use conjugate and diffuse priors for the coefficients and the variance covariance matrix, resulting in a multivariate normal posterior for the coefficients and an inverted Wishart posterior for the variance covariance matrix. For the prior for the coefficients restricted to zero we assume the mean to be zero and variance to be close to zero.

Our Gibbs sampler has the following structure. We first initialize the sampler with values for the coefficients and the variance covariance matrix. Conditional on data, the most recent draw for the coefficients and for the variance covariance matrix, we draw the latent state variables $\boldsymbol{\alpha}_t$ for $t = 1, ..., T$ using the procedure described in Carter and Kohn (1994). In the next step, we condition on data, the most recent draw for the latent variable $\boldsymbol{\alpha}_t$ and for the variance covariance matrix, drawing the coefficients from a multivariate normal distribution. Finally, conditional on data, the most recent draw for the latent variables and the coefficients, we draw the variance covariance matrix from an inverted Wishart distribution. We cycle through 100K Gibbs iterations, discarding the first 90K as burn-in. Of those 10K draws we save only every 10th draw, which gives us in total 1000 draws on which we base our inference. Convergence of the sampler was checked

[10]See Fixler et al. (2014) for a more detailed discussion of the GDE-GDI vintage history.

by studying recursive mean plots and by varying the starting values of the sampler and comparing results.

4 Results

To distinguish between the true unknown values ($\tilde{y}_t$) of GDP and our model's estimates of these values, we refer to our model's estimates as GDP^{++}. We compare our measure of GDP^{++} to other measures of GDP using graphs and historical decompositions, as well as by their dynamics and revision properties. We also examine how various releases of GDI and GDE correlate with GDP^{++} as well as with two other "reconciled" measures of GDP: the GDP^{+} measure produced by the Federal Reserve Bank of Philadelphia, and the simple average of GDE and GDI growth ($GDP^{50/50}$) published by the BEA.

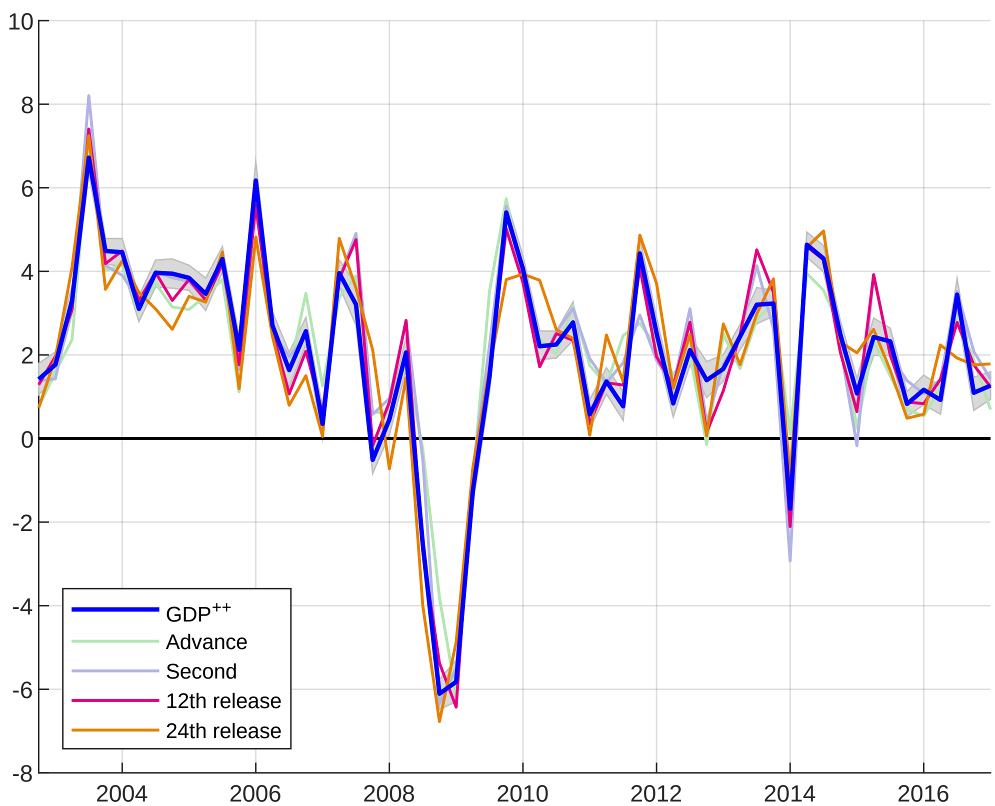
4.1 Comparison of GDP^{++} and releases of GDE and GDI

In Figure 3, we compare GDP^{++} and its shaded posterior ranges (90% of probability mass) to the four releases of GDE we employed in the estimation: the Advance, third, the 12th and the 24th release. There is little evidence that the releases are more volatile than GDP^{++}, which suggests that noise-type measurement errors are limited. On the other hand, we observe that the releases are outside the posterior bounds for some periods, particularly for the Advance release and the 24th release; in some periods, such as early 2007 and late 2008, the Advance release and the 24th release are on different sides of the posterior range.

Figure 4 shows GDP^{++} together with shaded posterior ranges (90% of probability mass) and the three releases of GDI we employed in the estimation: the Second/Third, the 12th and the 24th release. The GDI releases are more volatile than our GDP^{++} estimates (and more volatile than the releases of GDE), which is consistent with the presence of more substantial noise-type measurement error. We also see that the releases are often outside the posterior bounds of the true values.

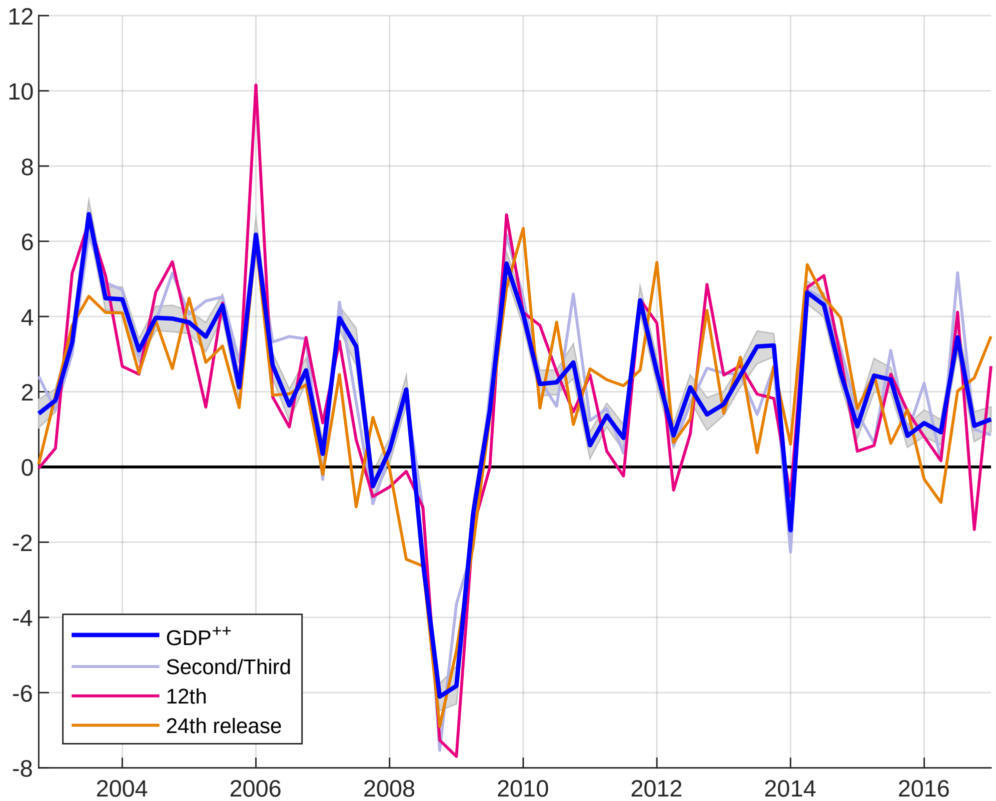
Moreover, Figure 10 and Figure 11 in the Appendix compare different releases of GDI and GDE with "Ragged Edge" estimates of GDP^{++}. The latter mimics the problem of estimating the previous quarter's GDP growth rate by using only the first two releases of

Figure 3: GDP^{++} vs. GDE



The blue line represents the posterior median of GDP^{++} (the smoothed estimate of the "true" value) and the shaded area around the blue line indicates 90% of posterior probability mass. The green line represents the advance estimate, the purple line is the second estimate, the red line the 12th release and the orange line the 24th release of expenditure-side GDP growth.

Figure 4: GDP^{++} vs. GDI



The blue line represents the posterior median of GDP^{++}, the smoothed estimate of the "true" value, and the shaded area around the blue line indicates 90% of posterior probability mass. The purple line is the second/third estimate, the red line the 12th release and the orange line the 24th release of income-side GDP growth.

GDE and the first release of GDI for the previous quarter, treating as missing observations those releases which are not yet available for other recent quarters, and using filtered rather than smoothed estimates of the state vector.[11] We find results very similar to those shown above, suggesting that revisions in GDP^{++} are relatively minor. We explore this further below in Sections 4.4 and 4.5.

Note that the sample paths of GDP^{+} and GDE and GDI in Aruoba et al. (2016, Figure 3) show a different picture than our Figures 3 and 4. Their measure tends to track GDI more closely than GDE, whereas ours does the opposite. Their result is surprising in the sense that the BEA has long advocated the use of expenditure-based estimates over income-based estimates, arguing that the underlying source data on expenditures were more complete and more reliable. Our model treats the two data sources as symmetric a priori, but arrives at the conclusion that GDE merits more weight in the reconciliation.

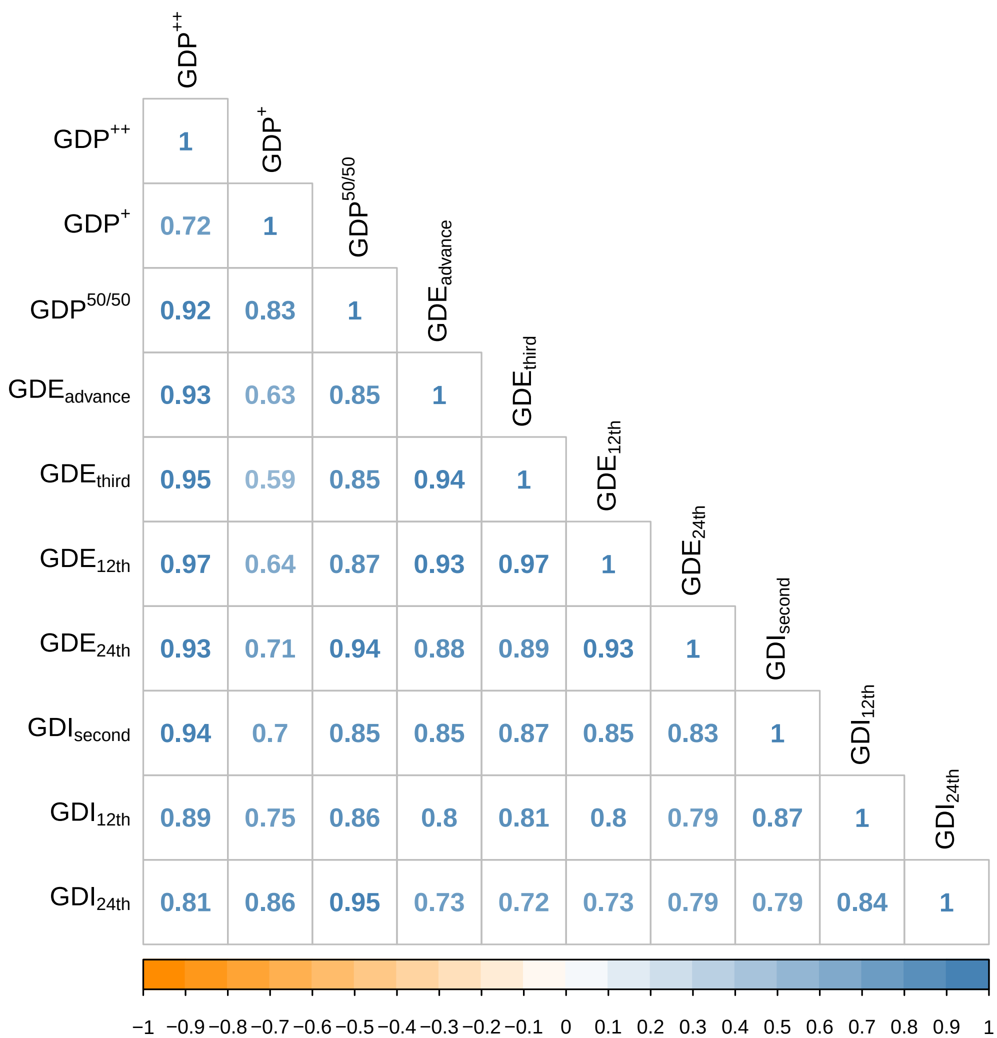
This can be seen more clearly in Figure 5, which reports contemporaneous correlations between various releases of GDE and GDI and the various estimates that attempt to reconcile the two. It shows that GDP^{+} is more tightly correlated with GDI releases than with the GDE corresponding releases, while the opposite is usually true for GDP^{++}. We also find that regardless of the series and the release chosen, GDP^{++} is almost always more positively correlated with the published series than GDP^{+}. (The only exception is the final release of GDI.) It is also more positively correlated with $GDP^{50/50}$. While there is no objective standard by which to judge that one measure of a latent variable (like "true" real GDP growth) is superior to another, we take comfort from the fact that our new estimate behaves more like the series that it is intended to reconcile, and that it puts more weight on the series that are generally acknowledged to be the more reliable.

4.2 Historical decomposition

Our econometric framework in (2) and (3) allows us to decompose each vintage of each of our series into its estimated news and noise measurement errors. The total revision of

[11] To estimate GDP^{++} with "Ragged Edge" data, we modify our Kalman filter equations to allow for missing observations; see, e.g., Durbin and Koopman (2001).

Figure 5: Correlations



Contemporaneous correlations between reconciled GDP measures and various releases of GDI and GDE. $GDP^{50/50}$ represents the mean of the latest vintage of GDE and GDI.

GDE and GDI can be written as

$$GDE_t^l - GDE_t^1 = \underbrace{-\nu_t^{GDE,1}}_{\text{News}} + \underbrace{\zeta_t^{GDE,l} - \zeta_t^{GDE,1}}_{\text{Noise}}, \tag{12}$$

$$GDI_t^l - GDI_t^1 = \underbrace{-\nu_t^{GDI,1}}_{\text{News}} + \underbrace{\zeta_t^{GDI,l} - \zeta_t^{GDI,1}}_{\text{Noise}}, \tag{13}$$

where every element on the right-hand side of the equation is part of the state vector which is estimated along with GDP^{++}.

These estimates are shown in Figure 6, with the top panel showing results for GDE and the bottom panel showing those for GDI. We see that total revisions in GDI tend to be larger than those in GDE (note the slight difference in vertical scales between the two panels.) We also observe that the news share in total GDE revisions tends to be larger than the noise share. While GDI revisions often incorporate substantial news, some of the largest revisions are due to noise, and noise errors seems to have a relatively more important role in GDI than in GDE. This observation is consistent with Fixler and Nailewaik (2009), who also reject the pure noise assumption in GDI. It also appears that GDI was particularly noisy in late 2007/early 2008 and around 2013.

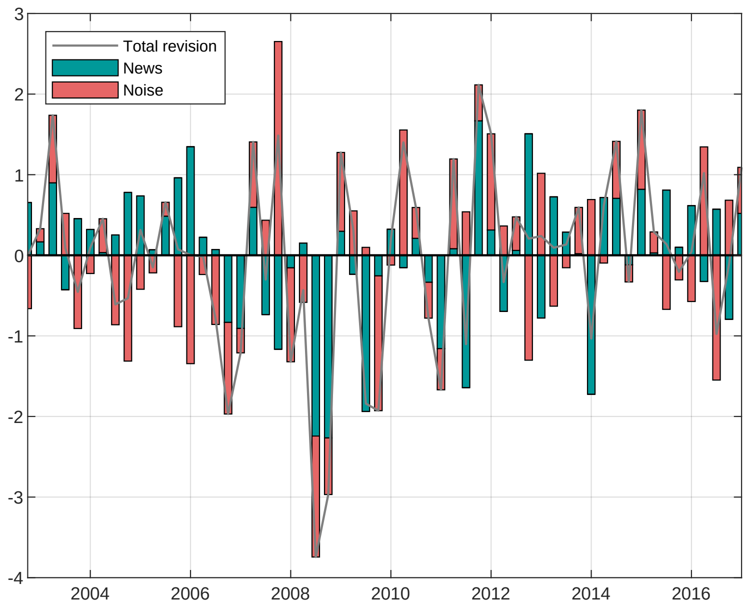
4.3 Comparing alternative measures of real output growth

To understand more about how the various measures of real GDP growth compare, Figure 7 compares them in terms of their persistence (measured by ρ, their first-order autocorrelation coefficient) and their variability (measured by σ^2). In light grey, we show the (ρ, σ^2) pairs for GDP^{++} across all draws, where $\sigma^2 \equiv r_1 \cdot r_1'$, where r_1 is the first row of $\boldsymbol{R}$ (as shown in (9)) so that σ^2 captures the innovation variance of shocks to the true values $\tilde{y}_t$. Against this background, we also show

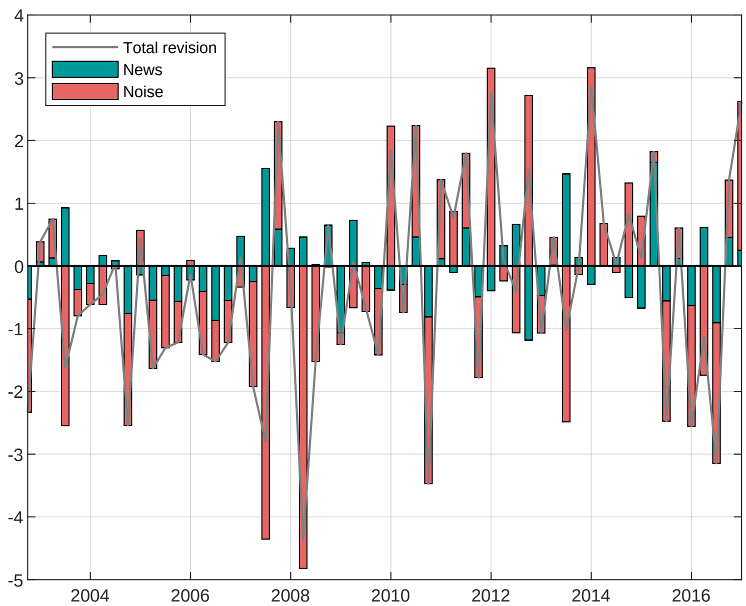
- the median estimate for GDP^{++},
- the estimates for AR(1) models fit to GDE and to GDI, respectively,
- the estimates for an AR(1) model fit to $GDP^{50/50}$,

Figure 6: Historical Decomposition of Total Revisions

GDE



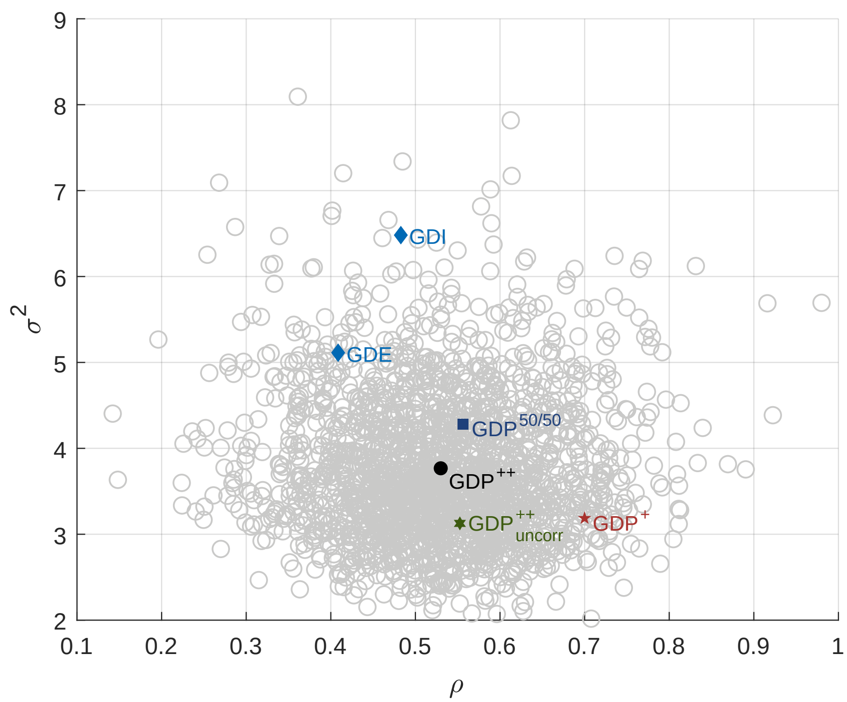
GDI



Historical decomposition of the total revision (24th release minus second estimate) into news and noise. The green bars depict the share of news and the red bars the share of noise in total revision (grey line). The historical decomposition is based on the decomposition described in (12) and (13).

- the estimates for the benchmark model estimated in Aruoba et al. (2016) (GDP^{+}),[12]
- the estimates for a restricted versions of our GDP^{++} model, where all measurement errors are assumed to be uncorrelated across GDE and GDI.

Figure 7: GDP Dynamics



The light grey shaded area consists of (ρ, σ^2) pairs across draws and the black circle is the posterior median of the (ρ, σ^2) pairs across draws from our sampler for the news-noise model with correlated measurement errors. The green six-pointed star depicts the posterior median of the (ρ, σ^2) pairs of the news-noise model without correlated measurement errors. The red five-pointed star is the posterior mean of the (ρ, σ^2) pairs of GDP^{+} using the benchmark specification ($\zeta = 0.8$) described in Aruoba et al. (2016). The blue diamonds are (ρ, σ^2) pairs, resulting from AR(1) models fitted to GDE and GDI, respectively. The dark blue square is the (ρ, σ^2) pair that results from an AR(1) model fitted to $GDP^{50/50}$. For GDP^{++} σ^2 is defined by $r_1 r_1'$, where r_1 is the first row of R defined in (9). The sampling period for re-estimating the Aruoba et al. (2016) model and for fitting the AR(1) models to the two GDP measures is 2002Q4–2017Q1 (released on May 05, 2019).

Figure 7 reveals that both our model and a version that restricts the measurement errors to be uncorrelated across the two series provide estimates with similar degrees of

[12]We thank Dongho Song for making his Matlab code available online.

persistence, although the restrictions decrease the innovation variances somewhat. GDE, GDI and $GDP^{50/50}$ all have degrees of persistence similar to that of GDP^{++} but considerably higher variance (especially GDE and GDI). GDP^{+} has by far the highest persistence, and a variance among the very lowest of those shown.

4.4 Relative contributions of GDE and GDI to GDP^{++}

To understand the relative importance of the different series and different releases to GDP^{++}, Table 1 presents the Kalman filter gains for each data release. The upper and lower panels of the Table show how weights change when we impose the restriction that measurement errors in GDE are uncorrelated with those in GDI. The "Balanced Sample" column shows the weights when all 7 data releases in our model are available, while the "Ragged Edge Sample" column shows how they change when only the first two releases of GDE and the first release of GDI are available for the most recent quarter.

Table 1: Kalman Gains

	Balanced Sample		Ragged-Edge Sample	
Weight on	GDE	GDI	GDE	GDI
News and Noise				
Advance	0.0272		0.2311	
Second/Third	-0.2103	0.3067	0.3363	0.4804
12th	0.7104	0.1081	0	0
24th Release	0.0479	0.0125	0	0
Uncorrelated News and Noise				
Advance	0.0380		0.1363	
Second/Third	0.1240	0.1672	0.4934	0.3768
12th	0.2318	0.0796	0	0
24th Release	0.2799	0.0826	0	0

Looking first at the "Ragged Edge" filter gains, we find that GDP^{++} initially puts almost as much weight on the first release of GDI as on the advance and next available release of GDE, something we might expect if GDE is thought to be noisy. Restricting the correlations of the measurement errors shifts some weight from GDI to GDE, with the most recent release of GDE receiving a weight of almost one-half.

When all the model's releases are available, however, GDP^{++} puts most of its weight on the 12th (but not the 24th) release of GDE and the initial (but not later) releases of GDI, suggesting that initial releases of GDI may contain important information that is somehow then lost as the BEA attempts to reconcile the two measures. We also see that allowing for correlated measurement errors across the two series is critical for this result; with uncorrelated errors, the pronounced weight on the initial release of GDI is greatly reduced and last two releases of GDE are roughly equally weighted.

4.5 Comparing revisions in GDP^{++}, GDP^{+} and $GDP^{50/50}$

Forecasters and policymakers are increasingly concerned about the revision properties of their data series when they rely on an estimate of recent economic conditions. For that reason, we examine the revisions in GDP^{++}, GDP^{+} and $GDP^{50/50}$ by comparing the "Ragged Edge" estimates analysed above with the "full-sample" filtered estimates that become available 22 months after the end of the quarter of the shortest sample. Due to limited availability of GDP^{+} vintages, our revision analysis is restricted to the samples $2002Q4 - 2013Q3 + i$ for $i = 0, ..., 22$.[13]

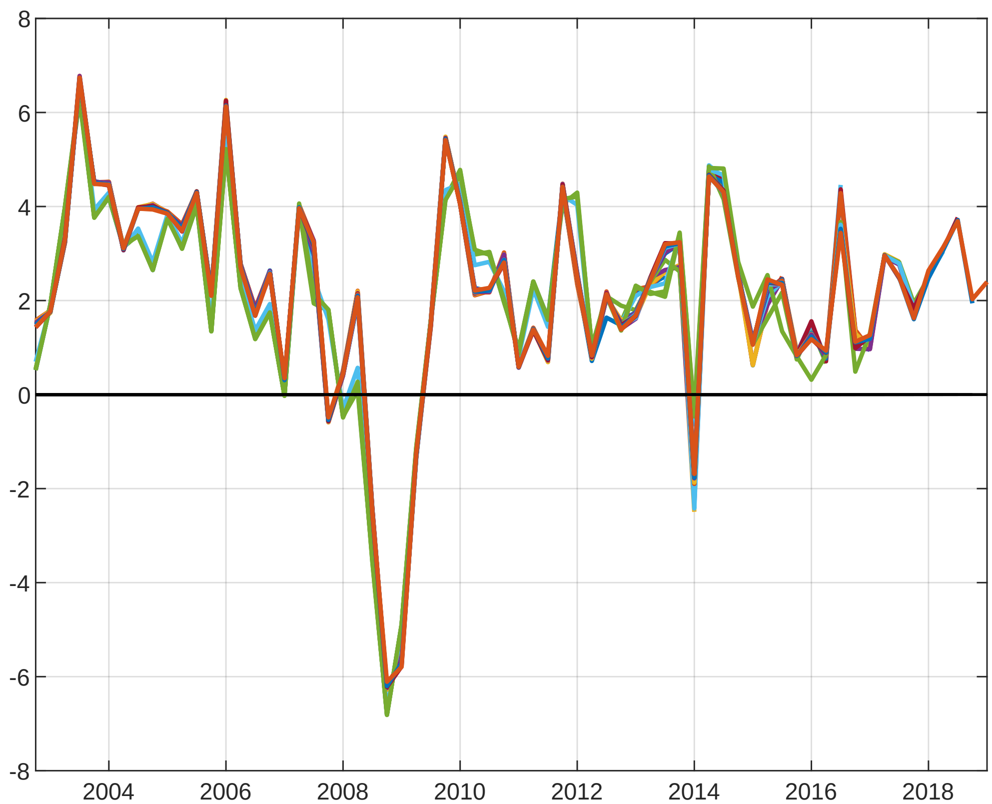
Figure 8 reports various vintages of GDP^{++} and shows that its revisions appear moderate. Moreover, most of the revisions seem to occur within the first couple of releases, but a few revisions reach far back to the beginning of the sample, reflecting possible benchmark revisions. Figure 9 compares the absolute value of the total revisions in GDP^{+}, GDP^{++} and $GDP^{50/50}$ for each period. According to Figure 9, GDP^{++} appears to be less prone to revisions than its competitors. This impression is confirmed by the root mean square of the revisions which is 0.39 for GDP^{++}, 1.20 for GDP^{+} and 1.15 for $GDP^{50/50}$.

5 Conclusion

We have described a new approach to data reconciliation that exploits multiple data releases on each series. This helps both with the identification of measurement errors and with optimally extracting information from multiple noisy but potentially informative series.

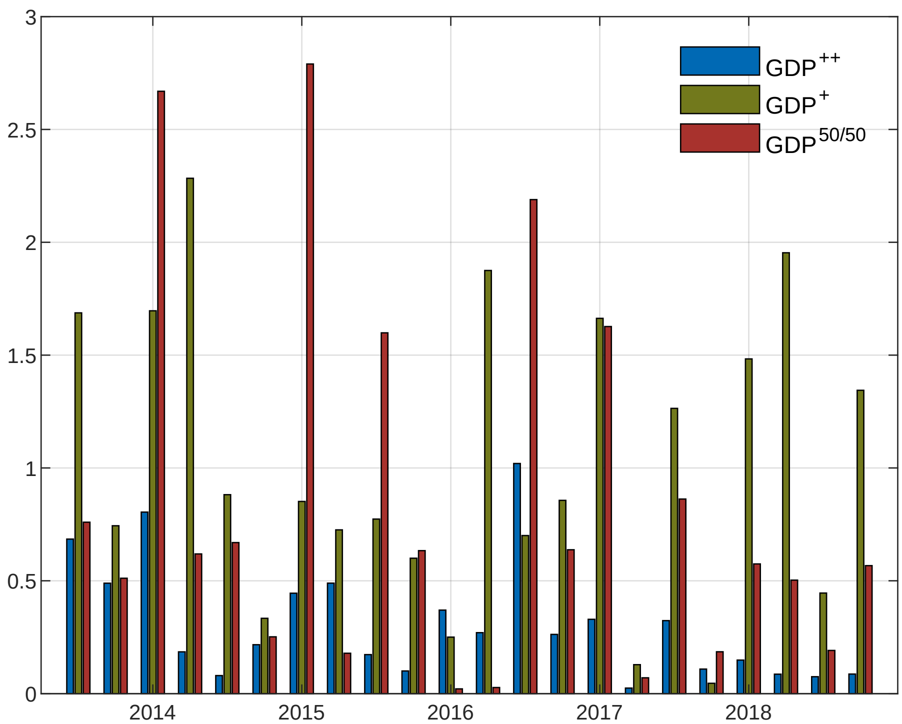
[13]The model was re-estimated for each sample $i = 0, ..., 23$, i.e., once for each quarter that we roll along.

Figure 8: GDP^{++} in Real-Time



Posterior medians of GDP^{++} from our news and noise model without correlated measurement errors for the samples $2002Q4 - 2013Q3 + i$ for $i = 0, ..., 22$. Each vintage of GDP^{++} is represented by a different colour.

Figure 9: Comparing Revisions in GDP^{++}, GDP^{+} and $GDP^{50/50}$



Absolute total revision in $GDP^{50/50}$, GDP^{+} and GDP^{++}. For GDP^{++} we used the posterior median of our news and noise model without correlated measurement errors, for GDP^{+} we employed the benchmark model of Aruoba et al. (2016) and $GDP^{50/50}$ is the mean of GDE and GDI. We have incorporated a missing observations approach as described in, e.g., Durbin and Koopman (2001) to cope with ragged edges at the end of the sample. The real-time analysis is based on the samples $2002Q4 - 2013Q3 + i$ for $i = 0, ..., 22$.

We used this to propose a new measure of U.S. GDP growth using multiple releases on GDE and GDI. Our measure GDP^{++} is shown to undergo smaller revisions on average than the GDP^{+} measure of Aruoba et al. (2016) or the simple average of GDE and GDI published by the BEA. GDP^{++} also puts more weight on expenditure-side estimates than either of these other measures, consistent with the finding that historical decompositions of GDE and GDI measurement errors reveal a larger news share in GDE than in GDI. Finally, our results point to the importance of smoothing measurement errors across multiple noisy releases.

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Appendix

A.1 Reformulating the model

To understand whether the state-space model is identified, we apply the conditions in Komunjer and Ng (2011). To follow and apply their proof, we adopt their notation for state-space models, which they write as[14]

$$X_{t+1} = A(\theta) \cdot X_t + B(\theta) \cdot \varepsilon_{t+1} \tag{A.1}$$

$$Y_{t+1} = C(\theta) \cdot X_t + D(\theta) \cdot \varepsilon_{t+1} \tag{A.2}$$

where $\theta \in \Theta$ is a real-valued parameter vector of length n_θ.

In the case of our model of GDE and GDI with l releases each, we have

$$Y_t \equiv \left[GDE_t^{1st}, GDE_t^{2nd}, \ldots, GDE_t^{l}, GDI_t^{1st}, GDI_t^{2nd}, \ldots, GDI_t^{l}\right]', \text{ a } 2l \times 1 \text{ column vector}$$

$$A(\theta) \equiv \rho, \text{ a scalar coefficient such that } 0 < |\rho| < 1$$

$$X_t \equiv \widetilde{y}_t, \text{ a scalar}$$

$$B(\theta) \equiv \left[\sigma_\nu^{E\prime}, \left(\sigma_\nu^{EI} + \sigma_\nu^{I}\right)', \mathbf{0}_{1\times 2l}\right], \text{ a } 1 \times 4l \text{ vector}$$

$$\varepsilon_t \equiv \text{a } 4l \times 1 \text{ column vector of i.i.d. random variables, jointly } \sim N(0, I)$$

$$C(\theta) \equiv \rho \cdot \iota_{2l\times 1}, \ (\iota \equiv \text{a column vector of 1's})$$

$$D(\theta) \equiv \begin{bmatrix} V\cdot \searrow\left(\sigma_\nu^{E}\right) & V\cdot \searrow\left(\sigma_\nu^{EI}\right) & \searrow\left(\sigma_\zeta^{E}\right) & \searrow\left(\sigma_\zeta^{EI}\right) \\ \mathbf{0}_{l\times l} & V\cdot \searrow\left(\sigma_\nu^{I}\right) & \mathbf{0}_{l\times l} & \searrow\left(\sigma_\zeta^{I}\right) \end{bmatrix}$$

$$V \equiv \begin{bmatrix} 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \\ \vdots & & \ddots & 0 \\ 1 & \ldots & \ldots & 1 \end{bmatrix}, \quad \begin{array}{l}\text{an } l \times l \text{ matrix with 0's above the main diagonal} \\ \text{and 1's everywhere else}\end{array}$$

$$\searrow(\sigma) \ \equiv \text{a diagonal matrix with the elements of } \sigma \text{ along the main diagonal}$$

$$\theta \equiv \left[\rho, \sigma_\nu^{E\prime}, \sigma_\nu^{EI\prime}, \sigma_\nu^{I\prime}, \sigma_\zeta^{E\prime}, \sigma_\zeta^{EI\prime}, \sigma_\zeta^{I\prime}\right] \text{ and each element of } \left\{\sigma_\nu^{E}, \sigma_\nu^{I}, \sigma_\zeta^{E}, \sigma_\zeta^{I}\right\} \text{ is } > 0$$

[14]See Komunjer and Ng (2011), equations 1a and 1b respectively.

$\sigma_\nu^E, \sigma_\nu^{EI}, \sigma_\nu^I, \sigma_\zeta^E, \sigma_\zeta^{EI}, \sigma_\zeta^I$ are each $l \times 1$ column vectors, giving $\theta 1 + 6l$ parameters in total.

A.2 Model Assumptions

Komunjer and Ng (2011) base their identification conditions on five underlying assumptions.[15]

1. $\forall\, \theta \in \Theta, t, s$

   (a) $E(\varepsilon_t) = 0$

   (b) $E(\varepsilon_t \cdot \varepsilon_s') = \delta_{t-s} \cdot \Sigma_\varepsilon(\theta)$
   where δ_{t-s} is a real-valued scalar and $\Sigma_\varepsilon(\theta)$ is positive definite with Cholesky decomposition $L_\varepsilon(\theta)$.

   As noted above, we define $\varepsilon_t \sim N(0, I)$ and i.i.d., in which case $\Sigma_\varepsilon(\theta) = I$ and $\delta_{t-s} = 1$ for $t = s$ and 0 otherwise.

2. $\forall\, \theta \in \Theta, z \qquad |z \cdot I - A(\theta)| = 0 \Longrightarrow |z| < 1$

   In our case, $A(\theta)$ is the scalar value ρ, so this assumption reduces to $|z - \rho| = 0 \Longrightarrow z = \rho$. Therefore, the above assumption will be correct so long as $|\rho| < 1$. We have instead imposed the slightly stricter assumption that $0 < |\rho| < 1$.

3. The mapping $\theta \longrightarrow \Lambda(\theta)$ is continuously differentiable.

   As we note below, this assumption is optional and will be not required for our proof.

4. $\forall\, \theta \in \Theta \qquad D(\theta) \cdot \Sigma_\varepsilon(\theta) \cdot D(\theta)'$ is non-singular.

[15] They classify our model as "non-singular" (because the dimension of ε is greater than that of Y), so assumptions 3, 4 and 5 below refer to their assumptions for "non-singular" models.

As noted for Assumption 1, in our case $\Sigma_\varepsilon(\theta) = I$, so this reduces to

$$
\begin{aligned}
&D(\theta) \cdot D(\theta)' = \\
&\qquad \begin{bmatrix} V\cdot \searrow(\sigma_\nu^E) & V\cdot \searrow(\sigma_\nu^{EI}) & \searrow(\sigma_\zeta^E) & \searrow(\sigma_\zeta^{EI}) \\ \mathbf{0}_{l\times l} & V\cdot \searrow(\sigma_\nu^I) & \mathbf{0}_{l\times l} & \searrow(\sigma_\zeta^I) \end{bmatrix} \\
&\qquad \cdot \begin{bmatrix} V\cdot \searrow(\sigma_\nu^E) & V\cdot \searrow(\sigma_\nu^{EI}) & \searrow(\sigma_\zeta^E) & \searrow(\sigma_\zeta^{EI}) \\ \mathbf{0}_{l\times l} & V\cdot \searrow(\sigma_\nu^I) & \mathbf{0}_{l\times l} & \searrow(\sigma_\zeta^I) \end{bmatrix}' \\
&= \begin{bmatrix} V\cdot \searrow\left(\sigma_\nu^{E^2} + \sigma_\nu^{EI^2}\right)\cdot V' + \searrow\left(\sigma_\zeta^{E^2} + \sigma_\zeta^{EI^2}\right) & V\cdot \searrow(\sigma_\nu^{EI})\cdot \searrow(\sigma_\nu^I)\cdot V' + \searrow(\sigma_\zeta^{EI})\cdot \searrow(\sigma_\zeta^I) \\ V\cdot \searrow(\sigma_\nu^I)\cdot \searrow(\sigma_\nu^{EI})\cdot V' + \searrow(\sigma_\zeta^{EI})\cdot \searrow(\sigma_\zeta^I) & V\cdot \searrow(\sigma_\nu^I)\cdot \searrow(\sigma_\nu^I)\cdot V' + \searrow\left(\sigma_\zeta^{I^2}\right) \end{bmatrix} \\
&= \begin{bmatrix} V\cdot \searrow\left(\sigma_\nu^{E^2} + \sigma_\nu^{EI^2}\right)\cdot V' & V\cdot \searrow(\sigma_\nu^{EI})\cdot \searrow(\sigma_\nu^I)\cdot V' \\ V\cdot \searrow(\sigma_\nu^I)\cdot \searrow(\sigma_\nu^{EI})\cdot V' & V\cdot \searrow(\sigma_\nu^I)\cdot \searrow(\sigma_\nu^I)\cdot V' \end{bmatrix} \\
&\quad + \begin{bmatrix} \searrow\left(\sigma_\zeta^{E^2} + \sigma_\zeta^{EI^2}\right) & \searrow(\sigma_\zeta^{EI})\cdot \searrow(\sigma_\zeta^I) \\ \searrow(\sigma_\zeta^{EI})\cdot \searrow(\sigma_\zeta^I) & \searrow\left(\sigma_\zeta^{I^2}\right) \end{bmatrix} \\
&= \Sigma_\nu + \Sigma_\zeta
\end{aligned}
$$

where Σ_ν is the $2l \times 2l$ variance-covariance matrix of the news errors and Σ_ζ is the $2l \times 2l$ variance-covariance matrix of the noise errors. Given that each element of $\left\{\sigma_\nu^E, \sigma_\nu^I, \sigma_\zeta^E, \sigma_\zeta^I\right\}$ is > 0, both Σ_ν and Σ_ζ will be Positive Definite.

5. In the case where X_t is scalar, they assume that

   (a) $K(\theta)$ has full row rank.

   We return to this assumption, below.

   (b) $C(\theta)'$ has full column rank.

   In our case $C(\theta)$ is $\rho \cdot \iota_{2l\times 1}$, so this assumption is already implied by Assumption 2.

Assumptions 3 and 5a relate to Komunjer and Ng (2011)'s reparameterisation of the system in the form[16]

$$\widehat{X}_{t+1|t+1} = A(\theta) \cdot \widehat{X}_{t|t} + K(\theta) \cdot a_{t+1} \tag{A.3}$$

$$Y_{t+1} = C(\theta) \cdot \widehat{X}_{t|t} + a_{t+1} \tag{A.4}$$

where

$$\begin{aligned}
\widehat{X}_{t|t} &\equiv \text{ the optimal linear predictor of } x_t \text{ given } \{Y_t, Y_{t-1}, \ldots\} \\
a_t &\equiv Y_{t+1} - C(\theta) \cdot \widehat{X}_{t|t}, \text{ the 1-step-ahead forecast error} \\
E(a_t \cdot a_t') &= \Sigma_a(\theta) \text{ with Cholesky decomposition } L_a(\theta) \\
K(\theta) &\equiv \text{the steady-state Kalman Gain} \\
\Lambda(\theta) &\equiv \left[\text{vec}(A(\theta))', \text{vec}(K(\theta))', \text{vec}(C(\theta))', \text{vech}(\Sigma_a(\theta))'\right]
\end{aligned}$$

They note that[17]

$$\Sigma_a(\theta) = C(\theta) \cdot \overline{\Sigma}(\theta) \cdot C(\theta)' + D(\theta) \cdot \Sigma_\varepsilon(\theta) \cdot D(\theta)' \tag{A.5}$$

$$K(\theta) = \left[A(\theta) \cdot \overline{\Sigma}(\theta) \cdot C(\theta)' + B(\theta) \cdot \Sigma_\varepsilon(\theta) \cdot D(\theta)'\right] \cdot \Sigma_a^{-1}(\theta) \tag{A.6}$$

where $\overline{\Sigma}(\theta)$ is the positive semi-definite solution to the discrete algebraic Riccati equation

$$\overline{\Sigma} = A \cdot \overline{\Sigma} \cdot A' \tag{A.7}$$

$$+ B \cdot \Sigma_\varepsilon \cdot B' - \left[A \cdot \overline{\Sigma} \cdot C' + B \cdot \Sigma_\varepsilon \cdot D'\right] \cdot \left[C \cdot \overline{\Sigma} \cdot C' + D \cdot \Sigma_\varepsilon \cdot D'\right]^{-1} \cdot \left[C \cdot \overline{\Sigma} \cdot A' + D \cdot \Sigma_\varepsilon \cdot B'\right]$$

$$= A \cdot \overline{\Sigma} \cdot A' + B \cdot \Sigma_\varepsilon \cdot B' - K \cdot \Sigma_a \cdot K' \tag{A.8}$$

(where we have temporarily dropped the dependence of all of these terms on θ to economise on notation.) We can simplify (A.5) and (A.6) somewhat using that facts that

- $A(\theta) \equiv \rho$
- $C(\theta) \equiv \rho \cdot \iota_{2l \times 1}$

[16] See their equations 9a and 9b.

[17] See Komunjer and Ng (2011), equations 18 and 19.

- $\overline{\Sigma}(\theta)$ is scalar.[18]

- $\Sigma_{\varepsilon}(\theta) = I$

to obtain

$$\Sigma_a(\theta) = \overline{\Sigma}(\theta) \cdot \rho^2 \cdot 1_{2l\times 2l} + D(\theta) \cdot D(\theta)' \tag{A.9}$$

$$K(\theta) = \left[\rho^2 \cdot \overline{\Sigma}(\theta) \cdot \iota_{1\times 2l} + B(\theta) \cdot D(\theta)'\right] \cdot \Sigma_a^{-1}(\theta) \tag{A.10}$$

Beginning with (A.9), we see that $\Sigma_a(\theta)$ is the sum of a positive semi-definite matrix and a positive-definite matrix, so the result must be positive-definite (implying that the variance of the one-step-ahead forecast errors is non-zero for all elements of Y_{t+1}.) This also implies that $\Sigma_a^{-1}(\theta)$ exists. Moving to (A.10), this also implies that $K(\theta)$ has full row rank whenever $\left[\rho^2 \cdot \overline{\Sigma}(\theta) \cdot \iota_{1\times 2l} + B(\theta) \cdot D(\theta)'\right]$ has full row rank. Because this is a $1 \times 2l$ row vector, this will satisfied whenever

$$\begin{aligned} 0_{1\times 2l} &\neq \left[\rho^2 \cdot \overline{\Sigma}(\theta) \cdot \iota_{1\times 2l} + B(\theta) \cdot D(\theta)'\right] \\ \text{or } \rho^2 \cdot \overline{\Sigma}(\theta) \cdot \iota_{1\times 2l} &\neq -B(\theta) \cdot D(\theta)' \end{aligned} \tag{A.11}$$

Using the fact that

$$\begin{aligned} B \cdot D' &= \left[\sigma_\nu^{E\prime}, \left(\sigma_\nu^{EI} + \sigma_\nu^I\right)', \mathbf{0}_{1\times 2l}\right] \cdot \begin{bmatrix} V \cdot \searrow\left(\sigma_\nu^E\right) & V \cdot \searrow\left(\sigma_\nu^{EI}\right) & \searrow\left(\sigma_\zeta^E\right) & \searrow\left(\sigma_\zeta^{EI}\right) \\ \mathbf{0}_{l\times l} & V \cdot \searrow\left(\sigma_\nu^I\right) & \mathbf{0}_{l\times l} & \searrow\left(\sigma_\zeta^I\right) \end{bmatrix}' \\ &= \left[\sigma_\nu^{E\prime}, \left(\sigma_\nu^{EI} + \sigma_\nu^I\right)', \mathbf{0}_{1\times 2l}\right] \cdot \begin{bmatrix} \searrow\left(\sigma_\nu^E\right) \cdot V' & \mathbf{0}_{l\times l} \\ \searrow\left(\sigma_\nu^{EI}\right) \cdot V' & \searrow\left(\sigma_\nu^I\right) \cdot V' \\ \searrow\left(\sigma_\zeta^E\right) & \mathbf{0}_{l\times l} \\ \searrow\left(\sigma_\zeta^{EI}\right) & \searrow\left(\sigma_\zeta^I\right) \end{bmatrix} \\ &= \left[\sigma_\nu^{E\prime} \cdot \searrow\left(\sigma_\nu^E\right) \cdot V' + \left(\sigma_\nu^{EI} + \sigma_\nu^I\right)' \cdot \searrow\left(\sigma_\nu^{EI}\right) \cdot V' \quad \left(\sigma_\nu^{EI} + \sigma_\nu^I\right)' \cdot \searrow\left(\sigma_\nu^I\right) \cdot V'\right] \\ &= \left[\left(\sigma_\nu^{E^2\prime} + \sigma_\nu^{EI^2\prime} + \sigma_\nu^{I\prime} \cdot \searrow\left(\sigma_\nu^{EI}\right)\right) \cdot V' \quad \left(\sigma_\nu^{EI\prime} \cdot \searrow\left(\sigma_\nu^I\right) + \sigma_\nu^{I^2\prime}\right) \cdot V'\right] \end{aligned} \tag{A.12}$$

[18]Komunjer and Ng (2011) note that assumptions 2 and 4-NS are sufficient to ensure that a solution to the Riccati equation exists.

we can rewrite (A.11) as

$$\rho^2 \cdot \overline{\Sigma}(\theta) \cdot \iota_{1\times 2l} \neq -\left[\left(\sigma_\nu^{E^2\prime} + \sigma_\nu^{EI^2\prime} + \sigma_\nu^{I\prime} \cdot \searrow \left(\sigma_\nu^{EI}\right)\right) \cdot V' \quad \left(\sigma_\nu^{EI\prime} \cdot \searrow \left(\sigma_\nu^{I}\right) + \sigma_\nu^{I^2\prime}\right) \cdot V'\right]$$

The left-hand side implies that every element of the right hand side must be equal to $\rho^2 \cdot \overline{\Sigma}(\theta) > 0$. However, V' is a full-rank matrix which takes cumulative sums; since the cumulative sums of a non-zero constant cannot themselves be constant, the condition will always be satisfied.

Therefore, we can conclude that Assumption 5a is implied by our previous assumptions.

A.3 Conditions for Identification

Komunjer and Ng (2011) provide two alternative propositions for identification in non-singular models such as ours. The first (Proposition 1-NS) requires only that Assumptions 1, 2, 4-NS and 5-NS are respected, while the second (Proposition 2-NS) requires that all five of the Assumptions described above are respected.[19]

A.3.1 Proposition 1-NS

Komunjer and Ng (2011) show that when Assumptions 1, 2, 4-NS and 5-NS hold, then θ_0 and θ_1 are observationally equivalent if and only if there exists a full rank $n_x \times n_x$ matrix T such that

$$A(\theta_1) = T \cdot A(\theta_0) \cdot T^{-1} \tag{A.13}$$

$$C(\theta_1) = C(\theta_0) \cdot T^{-1} \tag{A.14}$$

$$K(\theta_1) = T \cdot K(\theta_0) \tag{A.15}$$

$$\Sigma_a(\theta_1) = \Sigma_a(\theta_0) \tag{A.16}$$

[19] See Komunjer and Ng (2011), p. 2008-2009.

However, the fact that $n_x = 1$ implies that T must be a non-zero real-valued scalar, while the fact that $A(\theta) = \rho$ means that (A.13) simplifies to

$$\rho_1 = T \cdot \rho_0 \cdot T^{-1} = \rho_0 \cdot T \cdot T^{-1} = \rho_0$$

This in turn implies that (A.14) simplifies to

$$\begin{aligned} \rho_1 \cdot \iota_{2l\times 1} &= \rho_0 \cdot \iota_{2l\times 1} \cdot T^{-1} \\ \therefore \rho_0 \cdot \iota_{2l\times 1} &= \rho_0 \cdot \iota_{2l\times 1} \cdot T^{-1} \\ \therefore 1 &= T \end{aligned}$$

This can be used to simplify (A.15) to

$$K(\theta_1) = K(\theta_0) \tag{A.17}$$

At this point, it is interesting to review the heuristic argument offered in the body of the paper; that by adding more vintages to our state-space model, the number of observable moments increases faster than the number of unknown parameters, thereby enabling identification of the model parameters when $l > 1$. Komunjer and Ng (2011) note via (A.3) and (A.4) that $\{A(\theta), C(\theta), K(\theta), \Sigma_a(\theta)\}$ capture all the observable moments of our series. Their Proposition 1-NS states that if these are enough to uniquely identify θ, then no two distinct values of θ will be observationally equivalent. In our case, we see that $A(\theta)$ uniquely determines $\rho \in \theta$ and that $C(\theta)$ provides no additional information on θ. That leaves $6l$ elements of θ to be identified by $K(\theta)$ and $\Sigma_a(\theta)$. $K(\theta)$ has $2l$ elements and $\Sigma_a(\theta)$ is a symmetric $2l \times 2l$ matrix which therefore has $2l \cdot (2l+1)/2 = 2l^2 + l$ distinct elements, so together they give $3l + 2l^2$ elements to identify $6l$ elements in θ. Obviously

$$3l + 2l^2 \geq 6l \quad \Longleftrightarrow \quad 2l^2 \geq 3l \quad \Longleftrightarrow \quad l \cdot (2l-3) \geq 0$$

This is sufficient to establish that the model cannot be identified when $l = 1$, but may be identified for $l \geq 2$.

We now consider whether $\exists \;\; \theta_1 \neq \theta_0 \mid \Sigma_a(\theta_1) = \Sigma_a(\theta_0)$ and $K(\theta_1) = K(\theta_0)$ given

$\rho_1 = \rho_0 = \rho$.

A.3.2 Conditions on $\Sigma_a(\theta)$

First we note that (A.16) and (A.9) implies

$$\Sigma_a(\theta_1) = \overline{\Sigma}(\theta_1) \cdot \rho^2 \cdot 1_{2l\times 2l} + D(\theta_1) \cdot D(\theta_1)'$$
$$= \overline{\Sigma}(\theta_0) \cdot \rho^2 \cdot 1_{2l\times 2l} + D(\theta_0) \cdot D(\theta_0)' = \Sigma_a(\theta_0)$$
$$\therefore \left[\overline{\Sigma}(\theta_1) - \overline{\Sigma}(\theta_0)\right] \cdot \rho^2 \cdot 1_{2l\times 2l} = \left[D(\theta_0) \cdot D(\theta_0)'\right] - \left[D(\theta_1) \cdot D(\theta_1)'\right] \quad \text{(A.18)}$$

However, $\left[\overline{\Sigma}(\theta_1) - \overline{\Sigma}(\theta_0)\right] \cdot \rho^2$ is scalar, which implies that all $l \cdot (2l+1)$ entries of $\left[D(\theta_0) \cdot D(\theta_0)'\right] - \left[D(\theta_1) \cdot D(\theta_1)'\right]$ must be identical and equal to $\left[\overline{\Sigma}(\theta_1) - \overline{\Sigma}(\theta_0)\right] \cdot \rho^2$. Now let $[]_{ij}$ represent the element in ith row and jth column of a matrix or vector, and let $d_j(\theta) \equiv \left[\sigma_\nu^E\right]_{j1}^2 + \left[\sigma_\nu^{EI}\right]_{j1}^2$, $f_j(\theta) \equiv \left[\sigma_\nu^I\right]_{j1} \cdot \left[\sigma_\nu^{EI}\right]_{j1}$, $g_j(\theta) \equiv \left[\sigma_\nu^I\right]_{j1}^2$ so that

$$\left[D(\theta) \cdot D(\theta)'\right]_{ij} =$$
$$\left[\sigma_\zeta^E\right]_{j1}^2 + \left[\sigma_\zeta^{EI}\right]_{j1}^2 + \sum_{k=1}^{j} d_k(\theta) \quad \forall\ i,j\ |i=j,\ 1 \le j \le l \quad \text{(A.19)}$$
$$\sum_{k=1}^{\min(i,j)} d_k(\theta) \quad \forall\, i,j \mid i \ne j,\ 1 \le i,j \le l \quad \text{(A.20)}$$
$$\left[\sigma_\zeta^I\right]_{j1}^2 + \sum_{k=1}^{j} g_k(\theta) \quad \forall\, i,j \mid i=j,\ l+1 \le j \le 2l \quad \text{(A.21)}$$
$$\sum_{k=1}^{\min(i,j)} g_k(\theta) \quad \forall\, i,j \mid i \ne j,\ l+1 \le i,j \le 2l \quad \text{(A.22)}$$
$$\left[\sigma_\zeta^I\right]_{j1} \cdot \left[\sigma_\zeta^{EI}\right]_{j1} + \sum_{k=1}^{\min(i,j)} f_k(\theta) \quad \forall\, i,j \mid |i-j| = l,\ 1 \le i,j \le 2l \quad \text{(A.23)}$$
$$\sum_{k=1}^{\min(i,j)} f_k(\theta) \quad \text{otherwise} \quad \text{(A.24)}$$

Note that in the special case where $l = 1$, (A.20) (A.21) and (A.24) do not apply because it will always be the case that $i = j$. When $l > 1$, however, since (A.18) implies that all entries of $\left[D(\theta_0) \cdot D(\theta_0)'\right] - \left[D(\theta_1) \cdot D(\theta_1)'\right]$ must be identical, (A.21) and (A.22) together imply that

$$\sum_{k=1}^{j} g_k(\theta_0) - \sum_{k=1}^{j} g_k(\theta_1) = \left[\sigma_\zeta^I(\theta_0)\right]_{j1}^2 - \left[\sigma_\zeta^I(\theta_1)\right]_{j1}^2 + \sum_{k=1}^{j} g_k(\theta_0) - \sum_{k=1}^{j} g_k(\theta_1)$$
$$\therefore \sigma_\zeta^{I^2}(\theta_0) = \sigma_\zeta^{I^2}(\theta_1) \tag{A.25}$$

so the l parameters in σ_ζ^I are identified. Similar reasoning applied to (A.23) and (A.24) shows that

$$\sigma_\zeta^I(\theta_0) \cdot \sigma_\zeta^{EI}(\theta_0) = \sigma_\zeta^I(\theta_1) \cdot \sigma_\zeta^{EI}(\theta_1)$$
$$\therefore \sigma_\zeta^{EI}(\theta_0) = \sigma_\zeta^{EI}(\theta_1) \tag{A.26}$$

and analogously (A.19) and (A.20) imply

$$\sigma_\zeta^{E^2}(\theta_0) + \sigma_\zeta^{EI^2}(\theta_0) = \sigma_\zeta^{E^2}(\theta_1) + \sigma_\zeta^{EI^2}(\theta_1)$$
$$\therefore \sigma_\zeta^{E^2}(\theta_0) = \sigma_\zeta^{E^2}(\theta_1) \tag{A.27}$$

so the $2l$ parameters in σ_ζ^{EI} and σ_ζ^E are also identified whenever $l > 1$.

In addition, (A.18) and (A.20) give us

$$\left[\overline{\Sigma}(\theta_1) - \overline{\Sigma}(\theta_0)\right] \cdot \rho^2 = \sum_{j=1}^{m} d_j(\theta_0) - \sum_{j=1}^{m} d_j(\theta_1) \qquad \forall\, m \mid 1 \leq m \leq l$$

Since this holds for $m = 1$, it follows that $\left[\overline{\Sigma}(\theta_1) - \overline{\Sigma}(\theta_0)\right] \cdot \rho^2 = d_1(\theta_0) - d_1(\theta_1)$. Subtracting this from the above equation gives

$$0 = \sum_{j=2}^{m} d_j(\theta_0) - \sum_{j=2}^{m} d_j(\theta_1) \qquad \forall\, m \mid 2 \leq m \leq l$$
$$\therefore d_j(\theta_0) = d_j(\theta_1) \qquad \forall\, j \mid 2 \leq j \leq l$$
$$\therefore \left[\sigma_\nu^E(\theta_0)\right]_{j1}^2 + \left[\sigma_\nu^{EI}(\theta_0)\right]_{j1}^2 = \left[\sigma_\nu^E(\theta_1)\right]_{j1}^2 + \left[\sigma_\nu^{EI}(\theta_1)\right]_{j1}^2 \qquad \forall\, j \mid 2 \leq j \leq l$$

(A.18) and (A.24) give us

$$\left[\overline{\Sigma}\left(\theta_1\right)-\overline{\Sigma}\left(\theta_0\right)\right]\cdot\rho^2=\sum_{j=1}^{m}f_j\left(\theta_0\right)-\sum_{j=1}^{m}f_j\left(\theta_1\right)\qquad\forall\, m\mid 1\leq m\leq l$$

$$\therefore f_j\left(\theta_0\right)=f_j\left(\theta_1\right)\qquad\forall\, j\mid 2\leq j\leq l$$

$$\therefore\left[\sigma_\nu^I\left(\theta_0\right)\right]_{j1}\cdot\left[\sigma_\nu^{EI}\left(\theta_0\right)\right]_{j1}=\left[\sigma_\nu^I\left(\theta_1\right)\right]_{j1}\cdot\left[\sigma_\nu^{EI}\left(\theta_1\right)\right]_{j1}\qquad\forall\, j\mid 2\leq j\leq l$$

and (A.18) and (A.22) give us

$$\left[\overline{\Sigma}\left(\theta_1\right)-\overline{\Sigma}\left(\theta_0\right)\right]\cdot\rho^2=\sum_{j=1}^{m}g_j\left(\theta_0\right)-\sum_{j=1}^{m}g_j\left(\theta_1\right)\qquad\forall\, m\mid 1\leq m\leq l$$

$$\therefore g_j\left(\theta_0\right)=g_j\left(\theta_1\right)\qquad\forall\, j\mid 2\leq j\leq l\tag{A.28}$$

$$\therefore\left[\sigma_\nu^I\left(\theta_0\right)\right]_{j1}=\left[\sigma_\nu^I\left(\theta_1\right)\right]_{j1}\qquad\forall\, j\mid 2\leq j\leq l\tag{A.29}$$

$$\therefore\left[\sigma_\nu^{EI}\left(\theta_0\right)\right]_{j1}=\left[\sigma_\nu^{EI}\left(\theta_1\right)\right]_{j1}\qquad\forall\, j\mid 2\leq j\leq l\tag{A.30}$$

$$\therefore\left[\sigma_\nu^E\left(\theta_0\right)\right]_{j1}^2=\left[\sigma_\nu^E\left(\theta_1\right)\right]_{j1}^2\qquad\forall\, j\mid 2\leq j\leq l\tag{A.31}$$

so an additional $3\cdot(l-1)$ parameters are also identified whenever $l>1$. That leaves only the three parameters $\left\{\left[\sigma_\nu^I\left(\theta\right)\right]_{11},\left[\sigma_\nu^{EI}\left(\theta\right)\right]_{11},\left[\sigma_\nu^E\left(\theta\right)\right]_{11}\right\}$. For them, (A.18) implies

$$\left[\sigma_\nu^I\left(\theta_0\right)\right]_{11}^2-\left[\sigma_\nu^I\left(\theta_1\right)\right]_{11}^2=\left[\sigma_\nu^I\left(\theta_0\right)\right]_{11}\cdot\left[\sigma_\nu^{EI}\left(\theta_0\right)\right]_{11}-\left[\sigma_\nu^I\left(\theta_1\right)\right]_{11}\cdot\left[\sigma_\nu^{EI}\left(\theta_1\right)\right]_{11}\tag{A.32}$$

and

$$\left[\sigma_\nu^I\left(\theta_0\right)\right]_{11}^2-\left[\sigma_\nu^I\left(\theta_1\right)\right]_{11}^2=\left[\sigma_\nu^E\left(\theta_0\right)\right]_{11}^2+\left[\sigma_\nu^{EI}\left(\theta_0\right)\right]_{11}^2-\left[\sigma_\nu^E\left(\theta_1\right)\right]_{11}^2-\left[\sigma_\nu^{EI}\left(\theta_1\right)\right]_{11}^2\tag{A.33}$$

This together with any element from (A.18) involving $\left[\overline{\Sigma}\left(\theta_1\right)-\overline{\Sigma}\left(\theta_0\right)\right]\cdot\rho^2$, such as

$$\left[\overline{\Sigma}\left(\theta_1\right)-\overline{\Sigma}\left(\theta_0\right)\right]\cdot\rho^2=\left[\sigma_\nu^I\left(\theta_0\right)\right]_{11}^2-\left[\sigma_\nu^I\left(\theta_1\right)\right]_{11}^2\tag{A.34}$$

gives us a system of three quadratic equations in the three remaining parameters, plus $\overline{\Sigma}\left(\theta\right)$. For the latter, we must turn to the Riccati Equation.

A.3.3 DARE

As we noted above, the Discrete Algebraic Riccati Equation may be written as

$$\overline{\Sigma} = A \cdot \overline{\Sigma} \cdot A' + B \cdot \Sigma_\varepsilon \cdot B' - K \cdot \Sigma_a \cdot K'$$

Given (A.16) and (A.17), this implies that

$$\begin{aligned} &\overline{\Sigma}(\theta_0) - A(\theta_0) \cdot \overline{\Sigma}(\theta_0) \cdot A(\theta_0)' - B(\theta_0) \cdot \Sigma_\varepsilon(\theta_0) \cdot B(\theta_0)' \\ &= \overline{\Sigma}(\theta_1) - A(\theta_1) \cdot \overline{\Sigma}(\theta_1) \cdot A(\theta_1)' - B(\theta_1) \cdot \Sigma_\varepsilon(\theta_1) \cdot B(\theta_1)' \end{aligned}$$

which we may simplify using $\Sigma_\varepsilon = I$ and $A = \rho$ to give

$$\begin{aligned} \overline{\Sigma}(\theta_0) \cdot \left(1-\rho^2\right) - B(\theta_0) \cdot B(\theta_0)' &= \overline{\Sigma}(\theta_1) \cdot \left(1-\rho^2\right) - B(\theta_1) \cdot B(\theta_1)' \\ \therefore \left[\overline{\Sigma}(\theta_0) - \overline{\Sigma}(\theta_1)\right] \cdot \left(1-\rho^2\right) &= B(\theta_0) \cdot B(\theta_0)' - B(\theta_1) \cdot B(\theta_1)' \\ &= \sum_{j=1}^{l} \left[\sigma_\nu^E(\theta_0)\right]_{j1}^2 + (\left[\sigma_\nu^{EI}(\theta_0)\right]_{j1} + \left[\sigma_\nu^I(\theta_0)\right]_{j1})^2 \\ &- \sum_{j=1}^{l} \left[\sigma_\nu^E(\theta_1)\right]_{j1}^2 + (\left[\sigma_\nu^{EI}(\theta_1)\right]_{j1} + \left[\sigma_\nu^I(\theta_1)\right]_{j1})^2 \end{aligned}$$

However, from the results above, we know that $\forall\ j \mid 2 \leq j \leq l$ the terms in the two summations will be identical, so this simplifies further to

$$\begin{aligned} &\left[\overline{\Sigma}(\theta_0) - \overline{\Sigma}(\theta_1)\right] \cdot \left(1-\rho^2\right) = \\ &\left(\left[\sigma_\nu^E(\theta_0)\right]_{11}^2 - \left[\sigma_\nu^E(\theta_1)\right]_{11}^2\right) + (\left[\sigma_\nu^{EI}(\theta_0)\right]_{11} + \left[\sigma_\nu^I(\theta_0)\right]_{11})^2 - (\left[\sigma_\nu^{EI}(\theta_1)\right]_{11} + \left[\sigma_\nu^I(\theta_1)\right]_{11})^2 \end{aligned}$$

We may use this to eliminate $\left[\overline{\Sigma}(\theta_0) - \overline{\Sigma}(\theta_1)\right]$ from (A.34) and write our system of three equations in three unknowns more compactly using the notation $s_{i,j} \equiv [\sigma_\nu^j(\theta_i)]_{11}$ for $j \in$

$\{I, EI, E\}, i \in \{0,1\}$ as

$$s_{0,I}^2 - s_{1,I}^2 = s_{0,I} \cdot s_{0,EI} - s_{1,I} \cdot s_{1,EI} \tag{A.35}$$

$$s_{0,I}^2 - s_{1,I}^2 = s_{0,E}^2 - s_{1,E}^2 + s_{0,EI}^2 - s_{1,EI}^2 \tag{A.36}$$

$$s_{0,I}^2 - s_{1,I}^2 = \frac{\rho^2}{\rho^2 - 1} \cdot \left[\left(s_{0,E}^2 - s_{1,E}^2\right) + \left(s_{0,EI} + s_{0,I}\right)^2 - \left(s_{1,EI} + s_{1,I}\right)^2\right] \tag{A.37}$$

(A.36) and (A.37) imply

$$\begin{aligned}
s_{0,I}^2 - s_{1,I}^2 = \frac{\rho^2}{\rho^2 - 1} &\cdot \left[\left(\left(s_{0,I}^2 - s_{1,I}^2\right) - \left(s_{0,EI}^2 - s_{1,EI}^2\right)\right) + \left(s_{0,EI} + s_{0.I}\right)^2 - \left(s_{1,EI} + s_{1,I}\right)^2\right] \\
&\therefore \left(s_{0,I}^2 - s_{1,I}^2\right) \cdot \left(\frac{\rho^2 - 1}{\rho^2} - 1\right) + \left(s_{0,EI}^2 - s_{1,EI}^2\right) \\
&= \left(s_{0,EI}^2 - s_{1,EI}^2\right) + 2 \cdot \left(s_{0,I} \cdot s_{0,EI} - s_{1,I} \cdot s_{1,EI}\right) + \left(s_{0,I}^2 - s_{1,I}^2\right) \\
&\therefore \left(s_{0,I}^2 - s_{1,I}^2\right) \cdot \left(\frac{\rho^2 - 1}{\rho^2} - 2\right) = 2 \cdot \left(s_{0,I} \cdot s_{0,EI} - s_{1,I} \cdot s_{1,EI}\right)
\end{aligned}$$

Combing this with (A.35) gives us

$$\begin{aligned}
s_{0,I} \cdot s_{0,EI} - s_{1,I} \cdot s_{1,EI} &= \kappa \cdot \left(s_{0,I} \cdot s_{0,EI} - s_{1,I} \cdot s_{1,EI}\right) \\
&\therefore 0 = (\kappa - 1) \cdot \left(s_{0,I} \cdot s_{0,EI} - s_{1,I} \cdot s_{1,EI}\right)
\end{aligned}$$

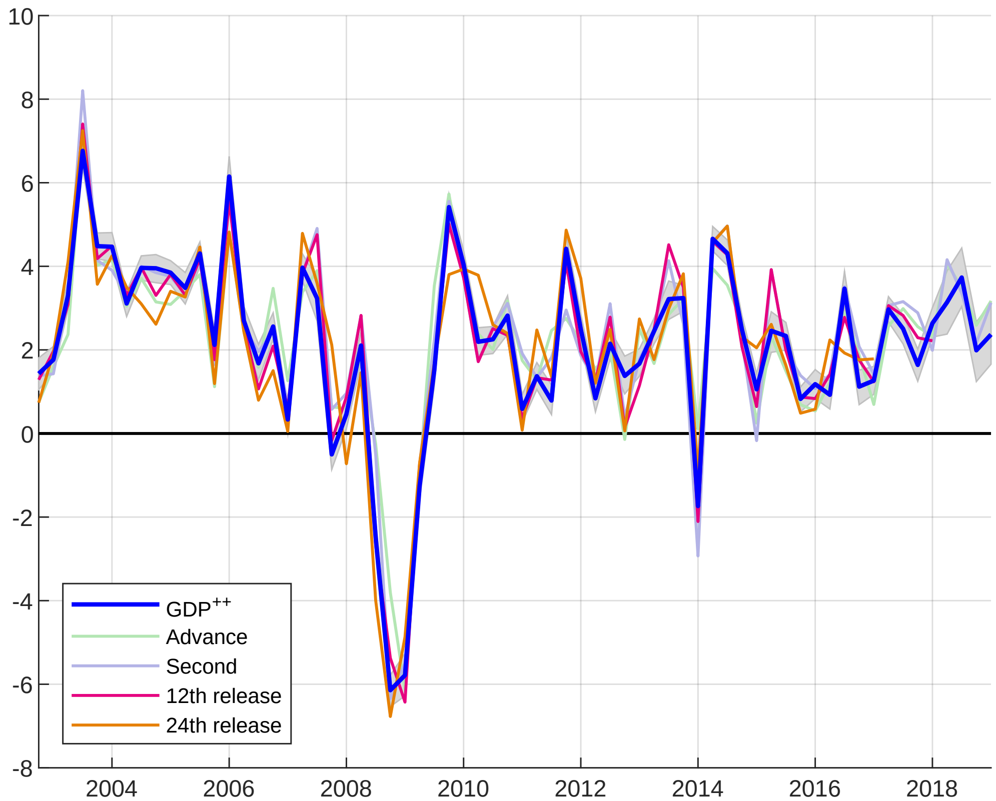
where $\kappa = 2 \cdot \left(\frac{\rho^2 - 1}{\rho^2} - 2\right)^{-1} = 2 \cdot \left(\frac{-\rho^2}{\rho^2 + 1}\right)$. Since ρ is real, this ensures that $\kappa \leq 0$, so $\kappa - 1 \neq 0$. The only solution is therefore

$$\begin{aligned}
s_{0,I} \cdot s_{0,EI} &= s_{1,I} \cdot s_{1,EI} \\
\therefore s_{0,I}^2 &= s_{1,I}^2 \ , \\
s_{0,EI} &= s_{1,EI} \ , \\
s_{0,E} &= s_{1,E}
\end{aligned}$$

which implies that all the remaining model parameters are identified.

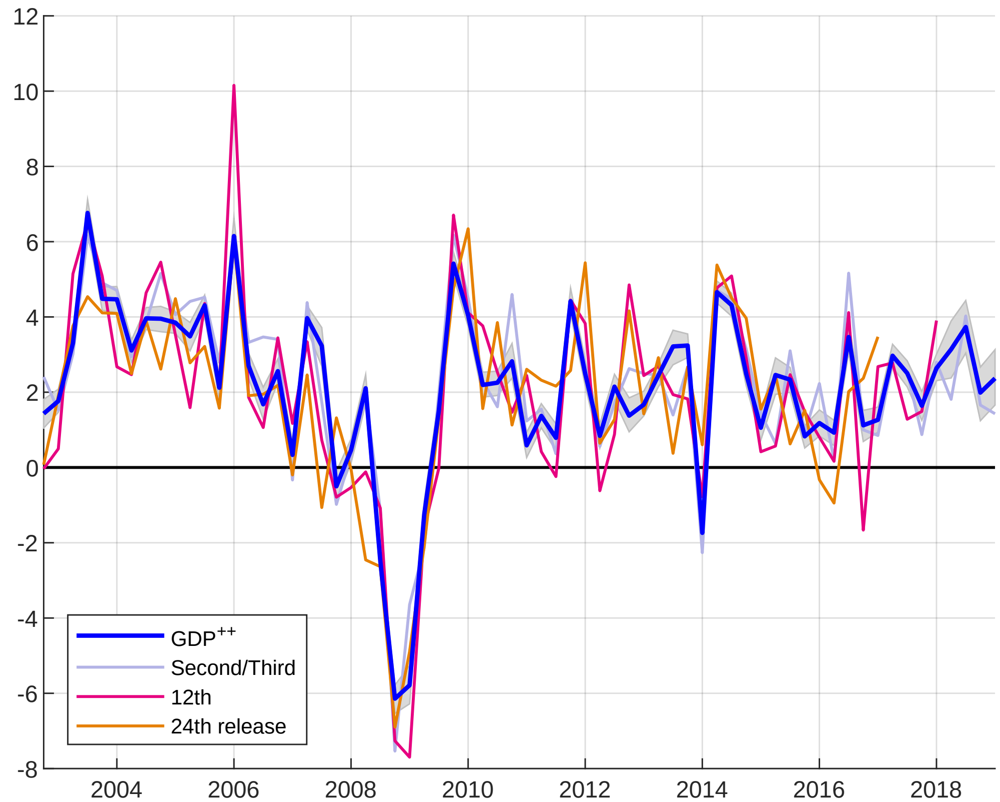
A.4 Additional figures

Figure 10: GDP^{++} vs. GDE with "Ragged-Edge" Data



The blue line represents the posterior median of GDP^{++} (the "true" value) and the shaded area around the blue line indicates 90% of posterior probability mass. The green line represents the advance estimate, the purple line is the second estimate, the red line the 12th release and the orange line the 24th release of expenditure side GDP growth. We have incorporated a missing observations approach as described in, e.g., Durbin and Koopman (2001) to cope with ragged edges at the end of the sample.

Figure 11: GDP^{++} vs. GDI with "Ragged-Edge" Data



The blue line represents the posterior median of GDP^{++}, the "true" value, and the shaded area around the blue line indicates 90% of posterior probability mass. The purple line is the second/third estimate, the red line the 12th release and the orange line the 24th release of income side GDP growth. We have incorporated a missing observations approach as described in, e.g., Durbin and Koopman (2001) to cope with ragged edges at the end of the sample.